

## AN IMPLICIT RUNGE-KUTTA METHOD FOR GENERAL SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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**Abstract.** This paper focuses on the derivation of a fully implicit Sixth order Runge-kutta type method with error estimation formula for the solution of general second order ordinary differential equations (ODEs). We define a transformation on the set of coefficients (Butcher coefficients Tableau) for first order differential equations to generate a new coefficient tableau for direct solution of second order ODEs. The scheme is simple, A-stable, highly efficient and has low implementation cost. Some problems are used as experimental examples.

### 1.0 Introduction

Development of Runge-kutta method for direct solution of second order Odes started with Nystrom [1]. He extended the fourth order explicit Runge-kutta method for first order ODEs to second order differential equations. Earlier researchers include Huta [2], Felhberg [3], Chawla et al [4], Sharp et al [5], Imoni et al [6] Coper et al [7], Filippi et al [8], Hairer [9] and Agam et al [10] etc. All these methods mentioned above are explicit type. Explicit Runge-kutta methods are unsuitable for solution of Stiff problems because their region of absolute stability is small, in particular they are bounded.

The instability of explicit methods motivates the development of implicit methods. Implicit Runge-kutta methods were earlier proposed by Kuntzmann [11] and Butcher [12]. Their proposed methods are based on Gauss quadrature. The remarkable thing about these methods are that their order  $p = 2s$ , for an S-stage scheme and are all A-stable [13]

The Gauss-quadrature method is a generalization of integrals in the interval  $(-1,1)$ . The Gauss-quadrature methods mentioned above are for first order differential equations. Thus there is need to develop parallel Runge-kutta methods for second and higher orders ODEs. In this paper we have tried to address these problems by deriving Runge-kutta method based on Gauss-quadrature for second order ODEs.

### 1.1 Preliminaries/ Basic definition

1.2 Definition: Let  $\langle G_1, * \rangle$  and  $\langle G_2, . \rangle$  be two set  $G_1$  and  $G_2$  with binary operations  $(*)$  and  $(.)$  respectively. A transformation  $T: G_1 \rightarrow G_2$  is a linear monomorphism if for any two points  $a_1, a_2 \in G_1$

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$$T(a_1 * a_2) = T(a_1) \cdot T(a_2) \text{ and } T(a_1) = T(a_2) \Rightarrow a_1 = a_2$$

Remark: A monomorphism operators preserve the algebraic structure of the domain into its co domain.

1.2 A Butcher Tableau of coefficients based on Gauss Legendre quadrature [14] of order 6, for solving first order ordinary differential equation is given below

**Table1 Butcher Coefficients table of order 6 for first order differential equations**

C	A		
$\left(\frac{1}{2} - \frac{\sqrt{15}}{10}\right)$	$\frac{5}{36}$	$\left(\frac{2}{9} - \frac{\sqrt{15}}{15}\right)$	$\left(\frac{5}{36} - \frac{\sqrt{15}}{30}\right)$
$\frac{1}{2}$	$\left(\frac{5}{36} + \frac{\sqrt{15}}{24}\right)$	$\frac{2}{9}$	$\left(\frac{5}{36} - \frac{\sqrt{15}}{24}\right)$
$\left(\frac{1}{2} + \frac{\sqrt{15}}{10}\right)$	$\left(\frac{5}{36} + \frac{\sqrt{15}}{30}\right)$	$\left(\frac{2}{9} + \frac{\sqrt{15}}{15}\right)$	$\frac{5}{36}$
$b^T$	$\frac{5}{18}$	$\frac{4}{9}$	$\frac{5}{18}$

C	A
$b^T$	$(b_1, b_2, \dots, b_s)$

where  $A = (a_{ij})$

$$C = (c_1, c_2, \dots, c_s)$$

$$b^T = (b_1, b_2, \dots, b_s) \tag{1.0}$$

1.4 Error of a Runge-kutta method is defined as

$$y(x_i) - y_i = (\text{Exact error})$$

Where  $y(x_i)$  is the exact solution at  $x = x_i$  and  $y_i$  is the RK solution at  $x = x_i$ .

### 2.0 Derivation method

We consider the general second order ODEs

$$a_2 y'' + a_1 y' + a_0 y = g(x, y), \quad y'(x_0) = y'_0 \tag{2.01}$$

where  $a_i, i = 0, 1, 2$  are scalar constants or functions. We then rewrite 2.01 in the form

$$y'' = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0 \tag{2.02}$$

$$= f(v), \quad v = (x, y, y')$$

We use the coefficient tableau (1.0)  $\in \mathbb{R}^3$  as our basis and domain of transformation T defined as follows.

$$v_i = \left( x + c_i h, y + \sum_{j=1}^3 a_{ij} T(v_j), y' + \sum_{j=1}^3 a_{ij} T'(v_j) \right) \in \mathbb{R}^3$$

$$T(v_i) = T\left(x + c_i h, y + \sum_{j=1}^3 a_{ij} T(v_j), y' + \sum_{j=1}^3 a_{ij} T'(v_j)\right) = h\left(y' + \sum_{j=1}^3 a_{ij} T'(v_j)\right)$$

and

$$T'(v_i) = hf\left(x + c_i h, y + \sum_{j=1}^3 a_{ij} T(v_j), y' + \sum_{j=1}^3 a_{ij} T'(v_j)\right) = h m_i$$

$$\text{i. e. } m_i = f\left(x + c_i h, y + \sum_{j=1}^3 a_{ij} T(v_j), y' + \sum_{j=1}^3 a_{ij} T'(v_j)\right) \quad (2.03)$$

**Theorem 1**

The Transformation  $T: \mathbb{R}^3 \rightarrow \mathbb{R}$  in (2.03) is a well defined monomorphism.

**Proof:**

let  $u, v \in \mathbb{R}$  defined by

$$U = \left(x + c_i h, y_1 + \sum_{j=1}^3 a_{ij} T(u_j), y'_1 + \sum_{j=1}^3 a_{ij} T'(u_j)\right)$$

$$V = \left(x + c_i h, y_2 + \sum_{j=1}^3 a_{ij} T(v_j), y'_2 + \sum_{j=1}^3 a_{ij} T'(v_j)\right)$$

by the definition of T on  $\mathbb{R}^3$ , we have

$$T(U + V) = h\left(y'_1 + \sum_{j=1}^3 a_{ij} T'(u_j) + y'_2 + \sum_{j=1}^3 a_{ij} T'(v_j)\right)$$

$$= h\left(y'_1 + \sum_{j=1}^3 a_{ij} T'(u_j)\right) + h\left(y'_2 + \sum_{j=1}^3 a_{ij} T'(v_j)\right)$$

$$= T(U) + T(V)$$

Hence T is a homomorphism.

Now we show that T is one to one

Let  $u, v \in \mathbb{R}^3$ , with  $T(u) = T(v)$ , then by definition of T, we have

$$h\left(y_1 + \sum_{j=1}^3 a_{ij} T'(u_j)\right) = h\left(y_2 + \sum_{j=1}^3 a_{ij} T'(v_j)\right)$$

$$\text{Since } T(u) = T(v) \text{ then } T'(u_j) = T'(v_j) \quad (2.04)$$

Hence  $y_1 = y_2$  and so

$$\left(x + c_i h, y_1 + \sum_{j=1}^3 a_{ij} T(u_j), y'_1 + \sum_{j=1}^3 a_{ij} T'(u_j)\right) = \left(x + c_i h, y_2 + \sum_{j=1}^3 a_{ij} T(v_j), y'_2 + \sum_{j=1}^3 a_{ij} T'(v_j)\right)$$

Hence  $U = V$

Thus  $T$  is one to one  $\Rightarrow$  monomorphism

Remark: By implication the domain of  $T$  and the image of  $T$  have the same algebraic structure.

We now use this transformation to generate new coefficients and Butcher tableau for direct solution of (2.02).

Using this transformation we have

$$\begin{aligned} T(v_1) &= h \left( y' + \sum_{j=1}^3 a_{1j} T(v_j) \right) = h \left( y' + \sum_{j=1}^3 a_{1j} h m_j \right) \\ &= h(y' + a_{11} h m_1 + a_{12} h m_2 + a_{13} h m_3), \text{ where } (a_{ij} \in \text{Table 1}) \\ &= h \left( y' + \frac{5}{36} h m_1 + \left( \frac{2}{9} - \frac{\sqrt{15}}{15} \right) h m_2 + \left( \frac{5}{36} - \frac{\sqrt{15}}{30} \right) h m_3 \right) \end{aligned}$$

Similarly,

$$\begin{aligned} T(v_2) &= T \left( x + c_2 h, y + \sum_{j=1}^3 a_{2j} T(v_j), y' + \sum_{j=1}^3 a_{2j} T'(v_j) \right) \\ &= h(y' + a_{21} h m_1 + a_{22} h m_2 + a_{23} h m_3) \\ &= h \left( y' + \left( \frac{5}{36} + \frac{\sqrt{15}}{24} \right) h m_1 + \frac{2}{9} h m_2 + \left( \frac{5}{36} - \frac{\sqrt{15}}{24} \right) h m_3 \right) \\ T(v_3) &= h(y' + a_{31} T'(v_1) + a_{32} T'(v_2) + a_{33} T'(v_3)) \\ &= h \left( y' + \left( \frac{5}{36} + \frac{\sqrt{15}}{30} \right) h m_1 + \left( \frac{2}{9} + \frac{\sqrt{15}}{15} \right) h m_2 + \frac{5}{36} h m_3 \right) \end{aligned}$$

By the definition of  $T$ , from (2.03)

$$m_i = T'(V_j) = f \left( x + c_i h, y + \sum_{j=1}^3 a_{ij} T(v_j), y' + \sum_{j=1}^3 a_{ij} T'(v_j) \right)$$

Now substituting for  $T(V_i)$  from (2.03), we have

$$\begin{aligned} m_1 &= (f(x + c_1 h, y + a_{11} T(v_1) + a_{12} T(v_2) + a_{13} T(v_3), y' + a_{11} T'(v_1) + a_{12} T'(v_2) \\ &+ a_{13} T'(v_3)) \end{aligned}$$

$$\begin{aligned} m_1 &= f \left( x + \left( \frac{1}{2} - \frac{\sqrt{15}}{10} \right) h, y + \frac{5}{36} h \left( y' + \sum_{j=1}^3 a_{ij} T(v_j) \right) + \left( \frac{2}{9} - \frac{\sqrt{15}}{15} \right) h \left( y' + \sum_{j=1}^3 a_{ij} T'(v_j) \right) \right) \\ &+ \left( \frac{5}{36} - \frac{\sqrt{15}}{30} \right) h \left( y' + \sum_{j=1}^3 a_{ij} T(v_j) \right), y' + \frac{5}{36} h m_1 + \left( \frac{2}{9} - \frac{\sqrt{15}}{15} \right) h m_2 + \left( \frac{5}{36} - \frac{\sqrt{15}}{30} \right) h m_3 \end{aligned}$$

$$(T'(v_i) = h m_i)$$

Simplifying and collecting like terms, we have

$$m_1 = f \left( x + \left( \frac{1}{2} - \frac{\sqrt{15}}{10} \right) h, y + \left( \frac{1}{2} - \frac{\sqrt{15}}{10} \right) hy' + \frac{1}{90} h^2 m_1 + \left( \frac{7}{90} - \frac{\sqrt{15}}{45} \right) h^2 m_2 \right. \\ \left. + \left( \frac{1}{9} - \frac{\sqrt{15}}{36} \right) h^2 m_3, y' + \frac{5}{36} hm_1 + \left( \frac{2}{9} - \frac{\sqrt{15}}{15} \right) hm_2 + \left( \frac{5}{36} - \frac{\sqrt{15}}{30} \right) hm_3 \right)$$

similarly

$$m_2 = f \left[ x + \frac{1}{2} h, y + a_{21} T(v_1) + a_{22} T(v_2) + a_{23} T(v_3), y' + a_{21} T'(v_1) + a_{22} T'(v_2) \right. \\ \left. + a_{23} T'(v_3) \right] \\ = f \left( x + \frac{1}{2} h, y + \left( \frac{5}{36} + \frac{\sqrt{15}}{24} \right) h \left( y' + \sum_{j=1}^3 a_{1j} hm_j \right) + \frac{2}{9} h \left( y' + \sum_{j=1}^3 a_{2j} hm_j \right) \right. \\ \left. + \left( \frac{5}{36} - \frac{\sqrt{15}}{24} \right) h \left( y' + \sum_{j=1}^3 a_{3j} hm_j \right), y' + a_{21} hm_1 + a_{22} hm_2 + a_{23} hm_3 \right)$$

$$m_2 = f \left( x + \frac{1}{2} h, y + \frac{1}{2} hy' + \left( \frac{7}{144} + \frac{\sqrt{15}}{72} \right) h^2 m_1 + \frac{1}{36} h^2 m_2 + \left( \frac{7}{144} - \frac{\sqrt{15}}{72} \right) h^2 m_3, \right. \\ \left. y' + \left( \frac{5}{36} + \frac{\sqrt{15}}{24} \right) hm_1 + \frac{2}{9} hm_2 + \left( \frac{5}{36} - \frac{\sqrt{15}}{24} \right) hm_3 \right)$$

$$m_3 = f \left( x + \left( \frac{1}{2} + \frac{\sqrt{15}}{10} \right) h, y + a_{31} T(v_1) + a_{32} T(v_2) + a_{33} T(v_3), y' + a_{31} T'(v_1) + a_{32} \right. \\ \left. + a_{33} T'(v_2) + a_{33} T'(v_3) \right) \\ = f \left( x + \left( \frac{1}{2} + \frac{\sqrt{15}}{10} \right) h, y + a_{31} h \left( y' + \sum_{j=1}^3 a_{1j} hm_j \right) + a_{32} h \left( y' + \sum_{j=1}^3 a_{2j} hm_j \right) \right. \\ \left. + a_{33} h \left( y' + \sum_{j=1}^3 a_{3j} hm_j \right), y' + \sum_{j=1}^3 a_{3j} hm_j \right)$$

Substituting for  $a_{ij}$  from table 1 and simplify, we have

$$m_3 = f \left( x + \left( \frac{1}{2} + \frac{\sqrt{15}}{10} \right) h, y + \left( \frac{1}{2} + \frac{\sqrt{15}}{10} \right) hy' + \left( \frac{1}{9} + \frac{\sqrt{15}}{36} \right) h^2 m_1 + \left( \frac{7}{90} + \frac{\sqrt{15}}{45} \right) h^2 m_2 \right. \\ \left. + \frac{1}{90} h^2 m_3, y' + \left( \frac{5}{36} + \frac{\sqrt{15}}{30} \right) hm_1 + \left( \frac{2}{9} + \frac{\sqrt{15}}{15} \right) hm_2 + \frac{5}{36} hm_3 \right)$$

The general solution of (2.02) is define as

$$y_{n+1} = y_n + b_1 T(v_1) + b_2 T(v_2) + b_3 T(v_3) \\ y_n + \frac{5}{18} h \left( y' + \frac{5}{36} hm_1 + \left( \frac{2}{9} + \frac{\sqrt{15}}{15} \right) hm_2 + \left( \frac{5}{36} - \frac{\sqrt{15}}{30} \right) hm_3 \right) + \frac{4}{9} h \left( y' + \left( \frac{5}{36} + \frac{\sqrt{15}}{24} \right) hm_1 + \right. \\ \left. \frac{2}{9} hm_2 + \left( \frac{5}{36} - \frac{\sqrt{15}}{24} \right) hm_3 \right) + \frac{5}{18} h \left( y' + \left( \frac{5}{36} + \frac{\sqrt{15}}{30} \right) hm_1 + \left( \frac{2}{9} + \frac{\sqrt{15}}{15} \right) hm_2 + \frac{5}{30} hm_3 \right)$$

(2.05)

$$\begin{aligned} y'_{n+1} &= y'_n + b_1 T'(v_1) + b_2 T'(v_2) + b_3 T'(v_3) \\ &= y'_n + \frac{5}{18} h m_1 + \frac{8}{18} h m_2 + \frac{5}{18} h m_3 \end{aligned}$$

Simplify (2.05), the general solution for solving (2.02) is

$$y_{n+1} = y_n + h y' + \frac{h^2}{36} [(5 + \sqrt{15})m_1 + 8m_2 + (5 - \sqrt{15})m_3]$$

$$y'_{n+1} = y'_n + \frac{5}{18} h [5(m_1 + m_3) + 8m_2]$$

where

$$\begin{aligned} m_1 &= f \left( x + \left( \frac{1}{2} - \frac{\sqrt{15}}{10} \right) h, y + \left( \frac{1}{2} - \frac{\sqrt{15}}{10} \right) h y' + \frac{1}{90} h^2 m_1 + \left( \frac{7}{90} - \frac{\sqrt{15}}{45} \right) h^2 m_2 \right. \\ &\quad \left. + \left( \frac{1}{9} - \frac{\sqrt{15}}{36} \right) h^2 m_3, y' + \frac{5}{36} h m_1 + \left( \frac{2}{9} - \frac{\sqrt{15}}{15} \right) h m_2 + \left( \frac{5}{36} - \frac{\sqrt{15}}{30} \right) h m_3 \right) \end{aligned}$$

$$m_2 = f \left( x + \frac{1}{2} h, y + \frac{1}{2} h y' + \left( \frac{7}{144} + \frac{\sqrt{15}}{72} \right) h^2 m_1 + \frac{1}{36} h^2 m_2 + \left( \frac{7}{144} - \frac{\sqrt{15}}{72} \right) h^2 m_3, y' + \left( \frac{5}{36} + \frac{\sqrt{15}}{24} \right) h m_1 + \frac{2}{9} h m_2 + \left( \frac{5}{36} - \frac{\sqrt{15}}{24} \right) h m_3 \right)$$

$$m_3 = f \left( x + \left( \frac{1}{2} + \frac{\sqrt{15}}{10} \right) h, y + \left( \frac{1}{2} + \frac{\sqrt{15}}{10} \right) h y' + \left( \frac{1}{9} + \frac{\sqrt{15}}{36} \right) h^2 m_1 + \left( \frac{7}{90} + \frac{\sqrt{15}}{45} \right) h^2 m_2 + \frac{1}{90} h^2 m_3, y' + \left( \frac{5}{36} + \frac{\sqrt{15}}{24} \right) h m_1 + \left( \frac{2}{9} + \frac{\sqrt{15}}{15} \right) h m_2 + \frac{5}{36} h m_3 \right)$$

### 3.0 Analysis of the scheme

The method for the second order ODEs is summarized in table 2:

**Table 2: Butcher's tableau for second order ODEs**

$C$	$A$			$A'$		
$\left( \frac{1}{2} - \frac{\sqrt{15}}{10} \right)$	$\frac{5}{36}$	$\left( \frac{2}{9} - \frac{\sqrt{15}}{15} \right)$	$\left( \frac{5}{36} - \frac{\sqrt{15}}{30} \right)$	$\frac{1}{90}$	$\left( \frac{7}{90} - \frac{\sqrt{15}}{45} \right)$	$\left( \frac{1}{9} - \frac{\sqrt{15}}{36} \right)$
$\frac{1}{2}$	$\left( \frac{5}{36} + \frac{\sqrt{15}}{24} \right)$	$\frac{2}{9}$	$\left( \frac{5}{36} - \frac{\sqrt{15}}{24} \right)$	$\left( \frac{7}{144} + \frac{\sqrt{15}}{72} \right)$	$\frac{1}{36}$	$\left( \frac{7}{144} - \frac{\sqrt{15}}{72} \right)$
$\left( \frac{1}{2} + \frac{\sqrt{15}}{10} \right)$	$\left( \frac{5}{36} + \frac{\sqrt{15}}{30} \right)$	$\left( \frac{2}{9} + \frac{\sqrt{15}}{15} \right)$	$\frac{5}{36}$	$\left( \frac{1}{9} + \frac{\sqrt{15}}{36} \right)$	$\left( \frac{7}{90} + \frac{\sqrt{15}}{45} \right)$	$\frac{1}{90}$
1	$\frac{5}{18}$	$\frac{8}{18}$	$\frac{5}{18}$	$\frac{1}{36} (5 + \sqrt{15})$	$\frac{8}{36}$	$\frac{1}{36} (5 - \sqrt{15})$
	$b^T$			$b'^T$		

$$\begin{array}{c|c} = C & AXA' \\ \hline & b^T X b'^T \end{array} = R(T) \in \mathfrak{R}^3$$

**Consistency:** Table 1 since the (Domain of T) is consistent, we concluded that table 2, is also consistent because it is the range of T. Since the  $\text{Dim}(T)$  is isomorphic to  $\text{Rang}(T)$  and T preserve the algebraic structure of Domain onto  $\text{Rang}(T)$

**Stability:** The test stability equation is

$$y'' = \lambda y, \quad \lambda < 0$$

For stability region, we set  $Z = \lambda h^2$

The stability function is defined as  $R(z)$  where

$$R(z) = I + (Zb^T + Z^2b'^T) (IZA - Z^2A')^{-1}e$$

$e = (1,1,1)^T$ ,  $h \in (0,1)$ ,  $D = A \times A'$  is the coefficient matrix, I is the identity 3 x

3 matrix,  $Z = \lambda h^2$ ,  $B = (b_1, b_2, b_3)$ ,  $b' = (b'_1, b'_2, b'_3)$  weights.

The A-stability Region is

$$(R(z)) = \{z: R(z) < 0 \text{ and } |R(z)| \leq 1\}$$

Our method is automatically A-stable since the  $\text{Dim}(T)$  based on Gauss quadrature method which is A-stable (see Butcher [4]), and T preserve its algebraic structure onto its Range.

**Error:** Since every Runge-kutta solution agree with Taylor's series expansion of order  $p = 6$

The error is  $C(h^{p+1}) = C(h^7)$

**Proposition 1 (Error estimation formula)**

If  $y_{n+1}^{(h)}$  and  $y_{n+1}^{(\frac{h}{2})}$  are approximate solutions of a Runge-kutta type method of order P with step size  $h$  and  $\frac{h}{2}$  respectively, then the error  $C(h^{p+1})$  is

$$E_r = \frac{2^{p+1}}{2^{p+1} - 1} \left[ y_{n+1}^{(\frac{h}{2})} - y_{n+1}^{(h)} \right]$$

**Proof**

Let  $y_{n+1}^{(h)}$  and  $y_{n+1}^{(\frac{h}{2})}$  be approximation solution of Runge-kutta method of order p respectively.

These solutions can be expanded into Taylor's series and Runge-kutta solutions agrees with the Taylor's series expansion up to the p terms and converges to exact solution  $y_{n+1}$ . Thus we can write

$$y_{n+1} = y_{n+1}^{(h)} + C(h^{p+1}) + R_n(x) \dots \dots (i) \quad (2.06)$$

$$y_{n+1} = y_{n+1}^{(\frac{h}{2})} + C\left(\frac{h}{2}\right)^{p+1} + Q_n(x) \dots \dots (ii) \quad (2.07)$$

as  $p \rightarrow \infty$ ,  $R_n(x)$  &  $Q_n(x) \rightarrow 0$ , hence  $R_n(x)$  &  $Q_n(x)$  can be ignored. Then subtracting (2.07) from (2.06) we have

$$0 = y_{n+1}^{(h)} - y_{n+1}^{(\frac{h}{2})} + C(h^{p+1}) \left( \frac{2^{p+1} - 1}{2^{p+1}} \right)$$

$$C(h^{p+1}) = \frac{2^{p+1}}{2^{p+1}-1} \left( y_{n+1}^{(\frac{h}{2})} - y_{n+1}^{(h)} \right)$$

Since  $C(h^{p+1})$  is the first neglect term of the series, our truncated error is  $C(h^{p+1})$ . Thus our error is

$$C(h^{p+1}) = E_r = \frac{2^{p+1}}{2^{p+1}-1} \left( y_{n+1}^{(\frac{h}{2})} - y_{n+1}^{(h)} \right)$$

Remark: Since our method is of order 6. The error of our Runge-kutta method is

$$E_r = \frac{2^7}{2^7-1} \left( y_{n+1}^{(\frac{h}{2})} - y_{n+1}^{(h)} \right) \quad (\text{error estimation formula}) \quad (2.08)$$

#### 4.0 Numerical Experiments

In this section we test the performances of our schemes by considering some problems with exact solutions to check quality, stability and implementation cost of our schemes

##### Example 1

$$y'' = x(y')^2, \quad y(0) = 1, y'(0) = \frac{1}{2}$$

$$\text{Analytic solution: } y(x) = 1 + \frac{1}{2} \ln \left( \frac{2+x}{2-x} \right)$$

##### Example 2

$$y'' - 3y' + 2y = x, \quad y(0) = 1, y'(0) = 0$$

$$\text{Analytic solution: } y(x) = e^x - \frac{3}{4}e^{2x} + \frac{1}{2}x + \frac{3}{4}$$

##### Example 3

$$y'' = -100y, \quad y(0) = 1, y'(0) = 10, h = 0.01$$

$$\text{Analytic solution: } y(x) = \cos(10x) + \sin(10x)$$

**Table 3: Comparison of the theoretical and approximate solutions for example 1**

$x$	$y(x)$	$y_{n+i}^{(h)}$	$y_{n+i}^{(\frac{h}{2})}$	Exact Errors $y(x_i) - y_{n+i}$	Approximation errors (2.08)
0.1	1.050041729278490	1.050041729277550	1.050041729278496	9.40 E (-13)	9.37 E (-13)
0.2	1.100335347731070	1.100335347729130	1.100335347731050	1.94 E (-12)	1.94 E (-12)
0.3	1.151140435936470	1.151140435933350	1.151140435936440	3.12 E (-12)	3.11 E (-12)
0.4	1.202732554054080	1.202732554049490	1.202732554054060	4.59 E (-12)	4.60 E (-12)
0.5	1.255412811883000	1.255412811876440	1.255412811882980	6.56 E (-12)	6.59 E (-12)



**Table 4: Comparison of the theoretical and approximate solutions for example 2**

$x$	$y(x)$	$y_{n+i}^{(h)}$	$y_{n+i}^{(\frac{h}{2})}$	Exact Errors $y(x_i) - y_{n+i}$	Approximation errors (2.08)
0.1	0.989118849455522	0.989118849340112	0.989118849453719	1.15 E (-10)	1.15 E (-10)
0.2	0.952534234929220	0.952534234647043	0.952534234924814	2.82 E (-10)	2.80 E (-10)
0.3	0.883269707283120	0.883269706765726	0.883269707275048	5.17 E (-10)	5.13 E (-10)
0.4	0.772669001271920	0.772669000428699	0.772669001258759	8.43 E (-10)	8.37 E (-10)
0.5	0.610009899355840	0.610009898067594	0.610009899335757	1.29 E (-09)	1.28 E (-09)

**Table 5: Comparison of the theoretical and approximate solutions for example 3**

$x$	$y(x)$	$y_{n+i}^{(h)}$	Error [15]	Exact Error
0.01	1.094837581924850	1.09483758192396	1.0 E (-08)	8.90E (-13)
0.02	1.178735908636300	1.17873590836347	2.3 E (-08)	1.55 E (-12)
0.03	1.250856695786950	1.25085669578490	4.0E (-08)	1.98 E (-12)
0.04	1.310479336311540	1.31047933630941	5.1 E (-08)	2.13 E (-12)
0.05	1.357008100494580	1.35700810049258	6.9 E (-08)	2.00E (-12)

## 5.0 Discussion of Results

We observed from the three problems tested the approximate error method (2.08) converges to the exact errors. This shows that our method is good and can be used to solve accurately problems without exact solutions. (see tables 3,4 and 5). The new method has only three function evaluations  $m_1, m_2, m_3$  which are directly used for integration of the problems. The method is much better than that of [15] (see table 5) in terms of accuracy and implementation cost.

## 6.0 Conclusion

We have used three Gauss-quadrature points to derived a 6<sup>th</sup> order implicit Runge-kutta method for direct integration of second order differential equations. The method is A-stable, highly efficient and has simple coefficients with low implementation cost. Alongside we develop an accurate error formula for step size control.

## Competing of Interests

Authors have declared that no competing interests exist.

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