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ON COMPLETENESS AND BICOMPLETIONS OF QUASI b-METRIC SPACES

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ABSTRACT. The purpose of this paper is to present results on completeness and bicompletions of quasi b-metric spaces. We further discuss extension of maps defined on a dense subspace to the whole space.

1. INTRODUCTION

In the literature one find a rich theory on completeness of metric spaces and quasi metric spaces for example: Doitchinov's completion [5], Yoneda's completion [10], Salbany's completion [13] and Smyth's completion [16]. Note that the current completion theory on metric spaces and quasi metric spaces does not cater and address for a completion of some mathematical structures: For example the space X, where X is the set of rational numbers equipped with the distance function $d(x,y) = (x-y)^2$. We easily see that (X,d) is neither a metric space nor a quasi metric space. The completion of X is just $\tilde{X}$, the set of real numbers with the distance function $d(x,y) = (x-y)^2$. So certainly, we need a formal and explicit structure to address this motivation.

The notion of b-metric spaces was first introduced and studied by Bakhtin [1] in 1989. Due to its importance, Czerwik used the notion to present and study some generalization of contraction mappings in [2] and also in [3]. Subsequently, several researchers motivated by the notion of b-metric spaces have obtained interesting extensions of the Banach contraction principle see the papers [4] and [11] for example.

Inspired by the notion of b-metric spaces we introduce the concept of quasi b-metric spaces and deduce several results in this context. Among other results presented in the paper, we show that every quasi b-metric space admits a bicompletion which is unique up to isometry, and we

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also show that every quasi b-metric space is quasi metrizable. On the other hand, we have provided some examples to show the generality of our results. In particular, we extend the work of [12] and some others.

2. Basic notions and preliminaries

Our basic references for quasi uniform spaces and quasi metric spaces are [6] and [9] respectively. Recall that:

Definition 2.1. [9] *A **quasi metric space** is a pair (X, d) where $d : X \times X \to [0, \infty)$ satisfies the following for all $x, y, z \in X$:*
(i) $d(x, y) = 0$ if and only if $x = y$.
(ii) $d(x, y) \leq d(x, z) + d(z, y)$.

Example 2.1. *Let $X = [0, \infty)$ and $d : X \times X \to [0, \infty)$ be defined by*

$$d(x,y) = \begin{cases} 1 & \text{if} \quad x > y, \\ 0 & \text{if} \quad x \leq y. \end{cases}$$

Then (X, d) is a quasi metric space.

Definition 2.2. [2] *Let X be a set, and $d : X \times X \to [0, \infty)$ be a function satisfies the following for all $x, y, z \in X$, and $s \geq 1$:*
(i) $d(x, y) = 0$ if and only if $x = y$.
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.
*(iii) $d(x, y) \leq s[d(x, z)+d(z, y)]$. The pair (X, d) is called a b-**metric space**.*

Example 2.2. *Let $X = \mathbb{R}$ and $d : X \times X \to [0, \infty)$ be defined by $d(x, y) = (x - y)^2$ for all $x, y \in \mathbb{R}$. Then (X, d) is a b-metric space with constant $(s = 2)$. Note that (X, d) is neither a quasi metric space nor a metric space.*

For more examples of b-metric spaces, which shows that b-metric space is a generalization of metric space we refer the reader to [7] and [14].

Remark 2.1. *Let (X, d) be a b-metric space. Note that:*
- *If $s = 1$ then, Definition 2.2 reduces to a standard metric space.*
- *If $s = 1$ and properties (i) and (iii) are satisfied then, Definition 2.2 reduces to a quasi metric space.*
- *Hence, the class of b-metric spaces is larger than the class of metric spaces.*

Definition 2.3. *A **quasi** b-**metric space** is a pair (X, d) where $s \geq 1, s \in \mathbb{R}$ and $d : X \times X \to [0, \infty)$ satisfies the following for all $x, y, z \in X$:*

(*i*) $d(x, y) = 0$ *if and only if* $x = y$.
(*ii*) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

Given a quasi b-metric space (X, d) we define $d^{-1} : X \times X \to [0, \infty)$ by $d^{-1}(x, y) = d(y, x)$ for all $x, y \in X$. Then (X, d^{-1}) is also a quasi b-metric space. We call d^{-1} the **conjugate** of d on X. Next, we define the function $d^* : X \times X \to [0, \infty)$ by $d^*(x, y) = max\{d(x, y), d^{-1}(x, y)\}$. Note that d^* is a b-**metric** on X.

Remark 2.2. *Note that a quasi metric space is a quasi b-metric space with a constant* $s = 1$. *Therefore the class of quasi b-metric spaces is larger than the class of quasi metric spaces.*

Definition 2.4. *Let* (X, d) *be a quasi b-metric space. Then:*
(*i*) *A sequence* $\{x_n\}$ *in* (X, d) d-***converges*** *to* $x \in X$ *if* $\lim_n d(x_n, x) = 0$.
(*ii*) *A sequence* $\{x_n\}$ *in* (X, d) d^{-1}-***converges*** *to a point* x *if* $\lim_n d^{-1}(x_n, x) = 0$.
(*iii*) *A sequence* $\{x_n\}$ *in* (X, d) d^*-***converges*** *to a point* x *if* $\lim_n d^*(x_n, x) = 0$.

Remark 2.3. *Let* (X, d) *be a quasi b-metric space. For a sequence* $\{x_n\}$ *in* X *and* $x \in X$, $\lim_n d^*(x_n, x) = 0$ *implies* $\lim_n d(x_n, x) = 0$ *and* $\lim_n d^{-1}(x_n, x) = 0$.

Example 2.3. *Let* $X = \mathbb{Q}$ *be equipped with* $d : X \times X \to [0, \infty)$ *defined by*

$$d(x, y) = \begin{cases} 1 & if \quad x > y, \\ (y - x)^3 & if \quad x \leq y. \end{cases}$$

Then (X, d) *is a quasi b-metric space but not a quasi metric space.*

3. Bicompletions of quasi b-metric spaces

Definition 3.1. *Let* (X, d) *be a quasi b-metric space. Then:*
(*i*) *A sequence* $\{x_n\}$ *in* (X, d) *is called a* d^*-***Cauchy*** *if* $\lim_{n,m} d^*(x_n, x_m) = 0$.
(*ii*) *A quasi b-metric space* (X, d) *is said to be* ***bicomplete*** *if every* d^**-Cauchy sequence* d^**-converges to a point* $x \in X$.

Remark 3.1. *A quasi b-metric space* (X, d) *is said to be bicomplete if* (X, d^*) *is a complete b-metric space.*

Definition 3.2. *Let* (X, d) *be a quasi b-metric space and* A *be a subset of* X. *Then* A *is* d^*-***dense*** *whenever it is dense in* X *with respect to* d^*.

Theorem 3.1. *Let (X, d) be a quasi b-metric space. Then every quasi b-metric space has a bicompletion.*

Proof: Let (X, d) be a quasi b-metric space and $\mathcal{C}$ be the set of all d^*-Cauchy sequences in (X, d). Define the relation $\sim$ on $\mathcal{C}$ as follows: $\{x_n\} \sim \{y_n\}$ if $\lim_n d(x_n, y_n) = 0$. It is easy to verify that $\sim$ is an equivalence relation in $\mathcal{C}$. Let $\tilde{X}$ be the set of all equivalence classes for $\sim$ and $\tilde{X} = \{[\{x_n\}] : x_n \in \mathcal{C}\}$. Define $\tilde{d} : \tilde{X} \times \tilde{X} \to [0, \infty)$ by $\tilde{d}([\{x_n\}], [\{y_n\}]) = \lim_n d(x_n, y_n)$.

We now show that $\tilde{d}$ is well defined. Let $\{x_n\}$ and $\{y_n\}$ be two d^*-Cauchy sequences in $\mathcal{C}$ then $\lim_{n,m} d^*(x_n, x_m) = 0$ and $\lim_{n,m} d^*(y_n, y_m) = 0$. Since,

$$d(x_n, y_n) \leq sd(x_n, x_m) + s^2 d(x_m, y_m) + s^2 d(y_m, y_n)$$

for $s \geq 1$. Then

$$d(x_n, y_n) - s^2 d(x_m, y_m) \leq sd(x_n, x_m) + s^2 d(y_m, y_n).$$

Note that

$$d(x_n, y_n) - d(x_m, y_m) \leq |d(x_n, y_n) - s^2 d(x_m, y_m)| \leq sd(x_n, x_m) + s^2 d(y_m, y_n).$$

Similarly, we have

$$d(x_m, y_m) - d(x_n, y_n) \leq |d(x_m, y_m) - s^2 d(x_n, y_n)| \leq sd(x_m, x_n) + s^2 d(y_n, y_m).$$

Hence,

$$|d(x_n, y_n) - d(x_m, y_m)| \leq sd(x_n, x_m) + s^2 d(y_m, y_n) \to 0.$$

Since $|d(x_n, y_n) - d(x_m, y_m)| \to 0$, this implies that $\{d(x_n, y_n)\}$ is a d^*-Cauchy sequences in $[0, \infty)$. Since $[0, \infty)$ is complete when equipped with the usual metric then, the limit of $\{d(x_n, y_n)\}$ exists. Thus $\tilde{d}$ is well defined.

We show that $\tilde{d}$ is a quasi b-metric on $\tilde{X}$. Let $[\{x_n\}], [\{y_n\}]$ and $[\{z_n\}] \in \tilde{X}$.

Suppose that $\tilde{d}([\{x_n\}], [\{y_n\}]) = 0$. Then $[\{x_n\}] = [\{y_n\}]$.

Conversely, suppose that $[\{x_n\}] = [\{y_n\}]$. Then

$$\tilde{d}([\{x_n\}], [\{y_n\}]) = \tilde{d}([\{x_n\}], [\{x_n\}]) = 0.$$

Therefore $\tilde{d}([\{x_n\}], [\{y_n\}]) = 0$.

We know that for all $x_n, y_n, z_n \in X$, $d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n)$. That is,

$$\begin{aligned}
\tilde{d}([\{x_n\}], [\{z_n\}]) = \lim_n d(x_n, z_n) &\leq \lim_n s[d(x_n, y_n) + \lim_n d(y_n, z_n)]. \\
&\leq s[\lim_n d(x_n, y_n) + \lim_n d(y_n, z_n)]. \\
&= s[\tilde{d}([\{x_n\}], [\{y_n\}]) + \tilde{d}([\{y_n\}], [\{z_n\}])].
\end{aligned}$$

Therefore $\tilde{d}$ is a quasi b-metric on $\tilde{X}$, with the same parameter s.

Let $f : X \to \tilde{X}$ be a map and be defined by $f(x) = \{\{x_n\} : x_n = x$ for all $n\}$. Since $\tilde{d}(f(x), f(y)) = \lim_n d(\{x_n\}, \{y_n\}) = \lim_n d(x_n, y_n) = d(x, y)$. Thus f is an isometry. f is injective since $\{\{x_n\} : x_n = x\} \neq \{\{y_n\} : y_n = y\}$ if $x \neq y$. Thus X can be regarded as a subspace of $\tilde{X}$.

Let $x = \{x_n\} \in \tilde{X}$. Given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n, m > N$, $d(x_n, x_m) < \epsilon$. In particular, $d(x_n, x_{N+1}) < \epsilon$ and $d(x_{N+1}, x_n) < \epsilon$ for $n > N$. Hence, $\tilde{d}(x, x_{N+1}) \leq \epsilon$ and $\tilde{d}(x_{N+1}, x) \leq \epsilon$. Thus X is d^*-dense in $\tilde{X}$.

Finally, we show that $(\tilde{X}, \tilde{d})$ is bicomplete. Let $x = \{x_n\}$ be a d^*-Cauchy sequence in X and $\epsilon > 0$. Then $d(x_n, x_m) < \epsilon$ for $m, n > N$. Fix $n > N$ and let $m \to \infty$ we get $\tilde{d}(x_n, x) < \epsilon, \tilde{d}(x, x_n) < \epsilon$. Hence $\tilde{d}(x_n, x) \to 0$ and $\tilde{d}(x, x_n) \to 0$ as $n \to \infty$.

Remark 3.2. *Let f be a map from a quasi b-metric space (X, d) into $(\tilde{X}, \tilde{d})$ such that for each $x \in X, f(x) = [\{\tilde{x}\}]$ where $[\{\tilde{x}\}]$ is the equivalence class of constant sequences. Then for each $x, y \in X$, $\tilde{d}(f(x), f(y)) = d(x, y)$, and hence, f is an isometry.*

Corollary 3.1. *Every b-metric space admits a completion.*

We now show by means of an example that Theorem 3.1 holds.

Example 3.1. *Let (X, d) be defined as in Example 2.3. Observe that (X, d) is not bicomplete. Its bicompletion is $\tilde{X} = \mathbb{R}$ with $\tilde{d} : \tilde{X} \times \tilde{X} \to [0, \infty)$ defined by*

$$\tilde{d}(x, y) = \begin{cases} 1 & \text{if} \quad x > y, \\ (y - x)^3 & \text{if} \quad x \leq y. \end{cases}$$

Note that both (X, d) and $(\tilde{X}, \tilde{d})$ have the same parameter.

4. Quasi b-metric spaces and quasi uniformities

Let (X, d) be a quasi b-metric space, then d on X induces a topology τ_d on X which has as a base the family of **open balls** $\{B_d(x, \epsilon) : x \in X, \varepsilon > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. In fact, we obtain a **bitopological space** $(X, \tau_d, \tau_{d^{-1}})$.

A topological space (X, τ) is called **quasi metrizable** if there is a quasi metric d on X such that $\tau = \tau_d$. In this case we say that d is compatible with τ, and that τ is a quasi metrizable topology.

Definition 4.1. [6] *A quasi uniformity on a set X is a filter $\mathcal{U}$ on $X \times X$ such that:*

(i) For each $U \in \mathcal{U}$, $\triangle \subset U$.

*(ii) If $U \in \mathcal{U}$, then $V \circ V \subset U$ for some $V \in \mathcal{U}$. The pair $(X,\mathcal{U})$ is called **quasi uniform space** and the members of $\mathcal{U}$ are called **entourages**.*

It worth noting that if $\mathcal{U}$ is a quasi uniformity on X, then $\mathcal{U}^{-1} = \{U^{-1}|U \in \mathcal{U}\}$ is also a quasi uniformity on X and is called the **conjugate** of $\mathcal{U}$ where $U^{-1} = \{(x,y) \in X \times X : (y,x) \in U\}$. A quasi uniformity $\mathcal{U}$ is a **uniformity** provided $\mathcal{U} = \mathcal{U}^{-1}$. For a quasi uniformity $(X,\mathcal{U})$, let $\mathcal{U}^* = \mathcal{U} \vee \mathcal{U}^{-1}$ then $\mathcal{U}^*$ is a uniformity on X, define $U[x] = \{y \in X : (x,y) \in U\}$. Then each quasi uniformity $\mathcal{U}$ on X induces a topology $\tau_{\mathcal{U}}$ on X, defined as follows: $\tau_{\mathcal{U}} = \{A \subset X :$ for each $x \in A$ there is $U \in \mathcal{U}$ such that $U[x] \subset A\}$.

Definition 4.2. [6] *A quasi uniform space $(X,\mathcal{U})$ is called **bicomplete** if $(X,\mathcal{U}^*)$ is a complete uniform space. In this case we say that $\mathcal{U}$ is a bicomplete quasi uniformity.*

It was proved in [6] that every quasi uniform space $(X,\mathcal{U})$ admits a unique bicompletion $(\tilde{X},\tilde{\mathcal{U}})$ up to quasi isomorphism.

Definition 4.3. *A quasi b-metric space (X,d) is **quasi uniformizable** if there exists a quasi uniform space $(X,\mathcal{U})$ such that $\tau_d = \tau_{\mathcal{U}}$.*

Theorem 4.1. *Every quasi b-metric space is quasi metrizable.*

Proof: For each $n \in \mathbb{N}$, define

$$U_n = \{(x,y) \in X \times X : d(x,y) < \frac{1}{n}\}.$$

We shall prove that $\{U_n\}$ is a base for the quasi uniformity on X whose induced topology coincides with τ_d. Note that for each $n \in \mathbb{N}$

$$\{(x,x) : x \in X\} \subseteq U_n, U_{n+1} \subset U_n.$$

For each $n \in \mathbb{N}$, choose $m \in \mathbb{N}$ such that $m > (s+2s^2)n$. Next we show that $U_m \circ U_m \circ U_m \subseteq U_n$. So let $(x,z) \in U_m \circ U_m \circ U_m \subseteq U_n$. We find $y,w \in X$ such that $(x,y) \in U_m$, $(y,w) \in U_m$ and $(w,z) \in U_m$. Then $d(x,y) < \frac{1}{m}$, $d(y,w) < \frac{1}{m}$ and $d(w,z) < \frac{1}{m}$. Hence,

$$\begin{aligned} d(x,z) &\leq sd(x,y) + s^2d(y,w) + s^2d(w,z) \\ &< \frac{2s^2+s}{m} \\ &< \frac{1}{n}. \end{aligned}$$

Therefore $(x,z) \in U_n$. Thus $\{U_n : n \in \mathbb{N}\}$ is a base for a quasi uniformity $\mathcal{U}$ on X. Therefore a quasi b-metric space is quasi uniformizable, we denote by $\mathcal{U}_d$ the quasi uniformity induced by d. Since for each $x \in X$ and each $n \in \mathbb{N}$,

$$\begin{aligned} U_n(x) &= \{y \in X : d(x,y) < \frac{1}{n}\} \\ &= B(x, \frac{1}{n}). \end{aligned}$$

Clearly, $\mathcal{U}_d$ has a countable base, hence (X, τ_d) is quasi metrizable [6] and [8]. In fact, $\tau_d = \tau_{\mathcal{U}_d}$. □

Corollary 4.1. *Every b-metric space is metrizable.*

Every quasi b-metric space (X, d) induces a quasi uniformity $(X,\mathcal{U}_d)$ as shown in Theorem 4.1. In addition, every quasi uniformity $(X,\mathcal{U}_d)$ has a bicompletion $(\tilde{X},\tilde{\mathcal{U}_d})$ [6].

Remark 4.1. *Given a quasi b-metric space (X, d), let $(\tilde{X}, \tilde{d})$ be the bicompletion of (X, d) as in Theorem 3.1 and $(\tilde{X},\mathcal{U}_{\tilde{d}})$ be the quasi uniformity of $(\tilde{X}, \tilde{d})$ as in Theorem 4.1. Further, let $(\tilde{X},\tilde{\mathcal{U}_d})$ be the bicompletion of $(X,\mathcal{U}_d)$, see, for instance,* [6].

Then:

Theorem 4.2. *Let (X, d) be a quasi b-metric space. Then, $(\tilde{X},\mathcal{U}_{\tilde{d}})$ is quasi isomorphic to $(\tilde{X},\tilde{\mathcal{U}_d})$. In particular, $\tau_{\mathcal{U}_{\tilde{d}}} = \tau_{\tilde{\mathcal{U}}_d}$.*

5. Extensions of maps on quasi b-metric spaces

Definition 5.1. *Let $f : (X, d) \to (Y, \rho)$ be a map between two quasi b-metric spaces. Then:*

*(i) A mapping $f : (X, d) \to (Y, \rho)$ is $(d$-$\rho)$-**continuous** if $\{x_n\}$ d-converges to x implies that $f(x_n)$ ρ-converges to $f(x)$.*

*(ii) A mapping $f : (X, d) \to (Y, \rho)$ is $(d^{-1}$-$\rho^{-1})$-**continuous** if $\{x_n\}$ d^{-1}-converges to x implies that $f(x_n)$ ρ^{-1}-converges to $f(x)$.*

*(iii) A mapping $f : (X, d) \to (Y, \rho)$ is $(d^*$-$\rho^*)$-**continuous** if $\{x_n\}$ d^*-converges to x implies that $f(x_n)$ ρ^*-converges to $f(x)$.*

Definition 5.2. *Let (X, d) and (Y, ρ) be two quasi b-metric spaces. A mapping $f : (X, d) \to (Y, \rho)$ is said to be **quasi uniformly continuous** if for any $\epsilon > 0$ there exists $\delta > 0$ such that $x, y \in X$ and $d(x, y) < \delta$ implies that $\rho(f(x), f(y)) < \epsilon$.*

Lemma 5.1. *Let (X, d) and (Y, ρ) be two quasi b-metric spaces. A mapping $f : (X, d) \to (Y, \rho)$ is quasi uniformly continuous if and only if $f : (X,\mathcal{U}_d) \to (Y,\mathcal{U}_\rho)$ is quasi uniformly continuous between the quasi uniform spaces.*

Proposition 5.1. *Let $a, d \in \mathbb{R}$ and $k \geq 1$. If $a \leq kd$ and $d \leq ka$ hold, then $a = d$.*

Theorem 5.1. *Let (X, d) be a quasi b-metric space and (Y, ρ) be a bicomplete quasi b-metric space. If $f : (A, d) \to (Y, \rho)$ is quasi uniformly continuous, from a d^*-dense subspace (A, d) of (X, d) to (Y, ρ), then there exists a unique extension $f^* : (X, d) \to (Y, \rho)$ such that $f^*|_A = f$ and f^* is quasi uniformly continuous. In particular, f^* is an isometry whenever f is so.*

Proof: By Lemma 5.1, f is quasi uniformly continuous from the quasi uniform space $(X, \mathcal{U}_d|_{A\times A})$ to the quasi uniform space $(Y, \mathcal{U}_\rho)$. Since A is d^*-dense in X, it follows that f has a unique quasi uniformly continuous extension $f^* : (X, \mathcal{U}_d) \to (Y, \mathcal{U}_\rho)$, see Theorem 3.29 in [6]. Thus f^* is quasi uniformly continuous from the quasi b-metric space (X, d) to the bicomplete quasi b-metric space (Y, ρ) by Lemma 5.1.

Let f be an isometry from (A, d) to (Y, ρ) and $x, y \in X$. Suppose that f is (d^*-ρ^*)-continuous. Then there exists $\{x_n\}$ and $\{y_n\}$ in A such that $\lim_n d^*(x_n, x) = 0$ and $\lim_n d^*(y_n, y) = 0$. By (d^*-ρ^*)-continuity of f^* it follows that $\lim_n \rho^*(f^*(x_n), f^*(x)) = 0$ and $\lim_n \rho^*(f^*(y_n), f^*(y)) = 0$. Then there exists $N \in \mathbb{N}$ such that
$d(x, x_n) < \epsilon, d(y, y_n) < \epsilon, \rho(f^*(x), f^*(x_n)) < \delta$ and $\rho(f^*(y), f^*(y_n)) < \delta$ for all $n \geq N$. Thus we have,

$$\begin{aligned} d(x,y) &\leq sd(x,x_n) + s^2 d(x_n,y_n) + s^2 d(y_n,y) \\ &\leq s\epsilon + s^2\rho(f^*(x_n), f^*(y_n)) + s^2\epsilon \\ &\leq s^2\rho(f^*(x_n), f^*(y_n)) \end{aligned}$$

and

$$\begin{aligned} \rho(f^*(x_n), f^*(y_n)) &\leq \alpha\rho(f^*(x_n), f^*(x)) \\ &+ \alpha^2\rho(f^*(x), f^*(y)) + \alpha^2\rho(f^*(y), f^*(y_n)) \\ &\leq \alpha\delta + \alpha^2\rho(f^*(x), f^*(y)) + \alpha^2\delta \\ &\leq \alpha^2\rho(f^*(x), f^*(y)) \end{aligned}$$

for all $n \in N$ and $\alpha \geq 1$. Therefore

$$d(x,y) \leq \alpha^2 s^2 \rho(f^*(x), f^*(y)) \qquad (1).$$

Similarly,

$$\begin{aligned} \rho(f^*(x),(f^*(y)) &\leq \alpha\rho(f^*(x),(f^*(x_n)) + \alpha^2\rho((f^*(x_n),(f^*(y_n)) \\ &+ \alpha^2\rho((f^*(y_n),(f^*(y)) \\ &\leq \alpha\delta + \alpha^2 d(x_n,y_n) + \alpha^2\delta \\ &\leq \alpha^2 d(x_n,y_n), \end{aligned}$$

and

$$\begin{aligned} d(x_n, y_n) &\leq sd(x_n, x) + s^2 d(x, y) + s^2 d(y, y_n) \\ &\leq s\epsilon + s^2 d(x, y) + s^2 \epsilon \\ &\leq \alpha^2 s^2 d(x, y) \end{aligned}$$

for all $n \in N$ and $s \geq 1$. Therefore

$$\rho(f^*(x), f^*(y)) \leq \alpha^2 s^2 d(x, y) \qquad (2).$$

By $(1), (2)$ and Proposition 5.1, we have $\rho(f^*(x), f^*(y)) = d(x, y)$. Hence f^* is an isometry from (X, d) to (Y, ρ). □

We see from Theorem 3.1 and Theorem 5.1 that: Every quasi b-metric space admits a bicompletion which is unique up to isometry. We now present the following results as Corollaries. Note that Corollary 5.1 and Corollary 5.3 are well known.

Corollary 5.1. *Let (X, d) be a quasi metric space and (Y, ρ) be a bicomplete quasi metric space. If $f : (A, d) \to (Y, \rho)$ is quasi uniformly continuous, from a d^*-dense subspace (A, d) of (X, d) to (Y, ρ), then there exists a unique extension $f^* : (X, d) \to (Y, \rho)$ such that $f^*|_A = f$ and f^* is quasi uniformly continuous. In particular, f^* is an isometry whenever f is so.*

Corollary 5.2. *Let (X, d) be a b-metric space and (Y, ρ) be a complete b-metric space. If $f : (A, d) \to (Y, \rho)$ is uniformly continuous, from a dense subspace (A, d) of (X, d) to (Y, ρ), then there exists a unique extension $f^* : (X, d) \to (Y, \rho)$ such that $f^*|_A = f$ and f^* is uniformly continuous. In particular, f^* is an isometry whenever f is so.*

Corollary 5.3. *Let (X, d) be a metric space and (Y, ρ) be a complete metric space. If $f : (A, d) \to (Y, \rho)$ is uniformly continuous, from a dense subspace (A, d) of (X, d) to (Y, ρ), then there exists a unique extension $f^* : (X, d) \to (Y, \rho)$ such that $f^*|_A = f$ and f^* is uniformly continuous. In particular, f^* is an isometry whenever f is so.*

We conclude the paper with the following example:

Example 5.1. *Let (X, d) be defined as in Examples 2.3 and 3.1. Consider $Y = \mathbb{R}$ be equipped with the usual metric. Define a mapping $f : (A, d) \to (Y, \rho)$ by $f(x) = x + 1$ for all $x \in A$. Then f is quasi uniformly continuous. Let $F : (X, d) \to (Y, \rho)$ be defined by $F(x) = x + 1$ for all $x \in X$, where $X = \mathbb{R}$. Then F extend f uniquely to X and it is quasi uniformly continuous. In particular, Theorem 5.1 holds.*

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