MAXIMUM PAIR-WISE DOMINATION NUMBER

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ABSTRACT. Let G be a connected, undirected graph without loops
and multiple edges. In network design graph models are extensively
used. Each vertex of a graph represents a computer in the network .
Data transition has a great importance in any network. In general
there are main server computers which keeps all data and send it to
other computers in network. It is quite important where to locate
the main server and how to take its backup. To decrease the loss of
data in any failure, the location of main server and backup should
be done carefully. This location process corresponds to another
type of domination in the graph model of the network. The vertex
which the main server is located should be adjacent to the vertex
which backup is located. The connection of these two vertices with
other vertices of the graph is expected to be maximum. For a pair
of vertices u and v the set of vertices which dominates both these
u and v is called a pairwise dominating set. The cardinality of this
set is called pairwise domination value of this uv pair. And among
all pair of vertices of the graph the maximum pairwise domination
value is called the maximum pair-wise domination number of G.

1. INTRODUCTION

Let $G = (V, E)$ be a graph of order n with no isolated vertex. Any
network can be modelled graphs where vertices and edges of G repre-
sent computers and links in a network respectively. The neighborhood
of a vertex $v \in V(G)$ is a set of vertices which are adjacent to v ,and
the open neighborhood of $N(v) = \{x \in V | vx \in E\}$, and the closed
neighborhood is $N[v] = N(v) \cup \{v\}$. In any network the number of the
vertices protecting (dominating) any pair of other vertices at the same
time is important for the security of the network and this number is
expected to be maximum. So the common neighborhood of any pair
of vertices has an importance in the security of data storage in any
network. In any graph G if any of two vertices are adjacent, then it

is said that they dominate each other. Domination number has been studied widely by graph theorists since many real life problems can be modelled under domination concept. This concept has been first examined by [1]. There are some basic domination parameters such as independent domination, total domination, connected domination, locating domination [2][3][4][5][6][7]. A dominating set of a graph G is a set of S of vertices such that every vertex of $V \backslash S$ is adjacent to some vertex in S. The domination number of G is the minimum cardinality of a dominating set and denoted by $\gamma(G)$. A paired dominating set of a graph $G = (V, E)$ with no isolated vertex is a dominating set of vertices including a graph with a perfect matching.[8][9] The paired domination number of G denoted by $\gamma_{pr}(G)$ is the minimum cardinality of a paired-dominating set of G.   In this paper, we define another parameter ***Maximum Pair-wise Domination Number***.

The Maximum Pair-wise Domination Number can be used in a problem which determines to place the main server and its backup server in any network system. Here the main purpose is to increase the number of communications between back up server and the rest of the working computers in any server failure. To make the communication strong between computers, the number of computers ,which are in direct communication with both main and backup server, should be increased. The storage units such as servers can be taken as the vertices of a graph and the communication links can be taken as the edges of a graph. Thus any communication network can be modelled by graphs. So that it can be seen by the information given above, the problem here is to find the maximum number of vertices both dominates these two vertices (main server and backup server) and to determine them.

Definition 1.   A spanning graph is a subgraph containing all the vertices of G.

Definition 2.   The complement of a graph G is the graph $\overline{G}$ with the same vertex set but whose edge set consists of the edges not present in G.

Definition 3.   A set $S \subset V$ is a dominating set if every vertex in V is either in S or is adjacent to a vertex in S, that is $V = \bigcup_{s \in S} N[s]$.

Definition 4.   The domination number $\gamma(G)$ is the minimum cardinality of a dominating set.

Definition 5. The closed neighborhood of a vertex $v \in V$ is denoted by $N[v] = N(V) \cup \{v\}$.

Definition 6. Maximum domination set is the set of elements which contains the intersection of neighborhoods of u and v vertices. In other words it contains the vertices which both dominates u and v at the same time.

$$MD = \{N(u) \bigcap N(v) \forall u, v \in V(G)\}$$

Definition 7. The average vertex degree of a connected graph $G = (V, E)$ is $2.E(G)/V(G)$.

Theorem 8. *For any disjoint u v vertices of graph G,*
$MD(u, v) \leq (deg(u) + deg(v))/2$

Proof. From the definition $MD = \{|N(u) \bigcap N(v)|\}$.
$|N(u) \bigcap N(v)| \leq N(u)$
$|N(u) \bigcap N(v)| \leq N(v)$
By elementary algebraic operations we can add both sides and the following result is obtained;
$|N(u) \bigcap N(v)| \leq (N(u) + N(v))/2$. □

2. MAXIMUM PAIR-WISE DOMINATION NUMBER AND SOME RESULTS

Definition 9. The Maximum Pair-wise Domination Number is the maximum cardinality of the dominating sets among all pair of vertices of the graph G and it is denoted by

$$MPDN = max\{|S| : S \text{ is the set of vertices in}$$
$$dom(u, v) \quad \forall u, v \in V(G)\}$$

Theorem 10. *For any connected graph G,*
$MDPN(P_n) \leq MDPN(G) \leq MPDN(K_n)$

Proof. In P_n the number of common vertices dominated by any vertex pair takes it least value since every nonadjacent vertex pair dominates only one vertex at most. For any K_n each vertex pair dominates $(n - 2)$ vertices except pair itself. So it is obvious. □

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs having the same order n with no isolated vertex.

Corollary 11. *If $\delta(G_1) > \delta(G_2)$ then $PW(G_1) > PW(G_2)$.*

Corollary 12. *If $\gamma(G_1) < \gamma(G_2)$ then $MPDN(G_1) < MPDN(G_2)$.*

Corollary 13. *If $|E(G_1)| < |E(G_2)|$ then $MPDN(G_1) < MPDN(G_2)$.*

Corollary 14. *If $average\, vertex\, degree\, of\, G_1 < average\, vertex\, degree\, of\, G_2$ then $MPDN(G_1) < MPDN(G_2)$.*

Theorem 15. *Let G be a connected graph and H be spanning subgraph of G.*
$MPDN(G) > MPDN(H)$

Proof. In any sub graph of a graph the number of edges decreases. This means that the number of common dominated vertices in any sub graph is smaller. Since the number of vertices in both G and H are the same , it is observed that $MDPN(H)$ value is smaller . □

Theorem 16. *Let G be a connected graph. $MPDN(G + e) > MPDN(G)$*

Proof. When an edge e is added between two disjoint vertices whose degrees are higher, the domination value of this vertex pair increases. So that the cardinality of maximum dominating set increases. □

Theorem 17. *Let G be a k-connected graph. $MPDN(G) \geq k$*

Proof. In any k-connected graph each u, v vertex pair has at least k common vertices dominated by both u and v. So $MPDN$ has at least value k. □

3. Maximum Pair-wise Domination Number of Basic Graphs

Result 1. For P_n Since every pair in a path can not have a common vertex more than 1 in their neighborhood $MPDN(P_n) = 1$.
Result 2. C_n,For $n > 4$ every vertex in a cycle lies on a path. Then $MPDN(C_n) = 1$.
Result 3. For complete graph K_n,so $MPDN(K_n) = n-1$ The number of neighbor vertices of a vertex pair has the maximum value in complete graph. It is known that every vertex has at most $n-1$ neighbors in K_n.
Result 4. For complete bipartite graph,$K_{m,n}$, $MPDN(K_{m,n}) = max\{m, n\}$
Result 5. For wheel graph $W_{1,n-1}$, $MPDN(W_{1,n-1}) = 2$
Result 6. For star graph $K_{1,n-1}$, $MPDN(K_{1,n-1}) = 1$
Result 7. For any n dimensional hypercube graph Q_n, $MPDN(Q_n) = 2$

4. GRAPH OPERATIONS AND MAXIMUM PAIR-WISE DOMINATION NUMBER

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with different orders and with no isolated vertex.

Definition 18. The join $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \bigcup G_2$ together with all the edges joining V_1 and V_2

Definition 19. The Cartesian product $G_1 * G_2$ of graphs G_1 and G_2 is a graph such that the vertex set of $G_1 * G_2$ is the Cartesian product $V(G_1) * V(G_2)$; and any two vertices (u, u') and (v, v') are adjacent in $G_1 * G_2$ if and only if either

$u = v$ and u' is adjacent with v' in G_2, or
$u' = v'$ and u is adjacent with v in G_1.

Result 1.

$$MDPN(G_1 + G_2) = \begin{cases} MDPN(G_1) + |V_2| & , MDPN(G_1) + |V_2| > MDPN(G_2) + |V_1| \\ MDPN(G_1) + |V_1| & , \text{otherwise} \end{cases}$$

Result 2.

$$MDPN(G_1 * G_2) = \begin{cases} MDPN(G_1)| & , if \quad G_1 > G_2 \\ MDPN(G_2)| & , if \quad G_2 > G_1 \end{cases}$$

5. AN ALGORITHM FOR FINDING MAXIMUM PAIR-WISE DOMINATION NUMBER OF A GRAPH

Our algorithm is based on the adjacency matrix of the graph. It uses the neighborhoods of vertices. "n" in const block represents the order of the graph. For any graph with order n Maximum Pair-wise Domination Number can be calculated as follows. The complexity of the algorithm is $O(n^3)$.

```
uses crt,graph;
const n=4 ;
var
PW : array[1..n*(n-1),1..n*(n-1)] of longint;
A : array[1..n,1..n] of longint;
PEN,k , i , j, u,v,t,max,p :longint;
BEGIN
```

```
for i:=1 to n do
   begin
      for j:=1 to n do
         begin
         readln(A[i,j]);
         end;
      end;
t:=0;
for i:=1 to (n-1) do
   begin
      for j:=i+1 to n do
         begin
         p:=0;
         t:=t+1;
          for k:=1 to n do
            begin
            if (A[i,k]=1) and (A[j,k]=1) then p:=p+1;
            end;
         PW[i,j]:=p; PW[j,i]:=p;
            if (t=1) or (max¡p) then begin
                     max:=p; u:=i; v:=j;
                     end;
         end;
end;
writeln('maximum pairwise domination number is',max);
write('the pair vertices are' u, v ,);
readln;
end.
```

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