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KUMARASWAMY LINDLEY-POISSON DISTRIBUTION: THEORY AND APPLICATIONS

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ABSTRACT. The Kumaraswamy Lindley-Poisson (KLP) distribution which is an extension of the Lindley-Poisson Distribution [21] is introduced and its properties are explored. This new distribution represents a more flexible model for the lifetime data. Some statistical properties of the proposed distribution including the shapes of the density and hazard rate functions are explored. Moments, entropy measures and the distribution of the order statistics are given. The maximum likelihood estimation technique is used to estimate the model parameters and a simulation study is conducted to investigate the performance of the maximum likelihood estimates. Finally some applications of the model with real data sets are presented to illustrate the usefulness of the proposed distribution.

1. Introduction

Lindley distribution [16], studied by Lindley in the context of fiducial and Bayesian statistics, is very useful for modeling failure time data. This distribution accommodates hazard rate functions that are increasing, decreasing or constant. However, models with complex hazard rate shapes such as unimodal, bathtub and other shapes are desirable in reliability analysis, human mortality studies and related areas. Ghitany et al. [11] proposed and studied the power Lindley distribution. Properties and applications of the Lindley distribution have been studied in the context of reliability analysis by several authors including Ghitany et al. [10], Sankaran [26] and Asgharzadeh et al. [1]. Nadarajah et al. [20] proposed and developed the mathematical properties of the generalized Lindley distribution. Properties of the exponentiated Power Lindley distribution were studied by Warahena-Liyanage and Pararai [35].

Several new families of distributions have been derived by compounding the Poisson distribution with many other continuous distributions to provide more flexible distributions for modeling lifetime data. Kuş [15] studied the exponential-Poisson distribution. Lu and Shi [17] derived and studied the Weibull-Poisson distribution. The exponentiated Weibull-Poisson distribution which generalizes the Weibull-Poisson was studied by Mahmoudi and Sepahdar [18], and the beta Weibull-Poisson was introduced and studied by Percontini et al.[23]. Barreto-Souza and Cribari-Neto [3] studied the exponentiated exponential-Poisson distribution. The two parameter Poisson-exponential distribution with an increasing failure rate was studied by Cancho et al.[5]. Recently,

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Pararai et al. [21] studied the properties of the exponentiated power Lindley-Poisson distribution, thereby generalizing the Lindley-Poisson distribution.

The properties of Kumaraswamy [14] distribution were explored in detail by Jones [13]. The author contrasted the Kumaraswamy distribution with the beta distribution. Some of the good properties of the Kumaraswamy distribution include a simple normalizing constant, closed form solutions of the distribution and quantile functions as well as simple formulas for the moments. Cordeiro et al. [7] studied the Kumaraswamy-Weibull distribution and applied the model to some failure data.

Motivated by the advantages of the generalized distribution with respect to having a hazard function that exhibits increasing, decreasing and bathtub shapes, as well as the versatility and flexibility of compounding Lindley and Poisson distributions in modeling lifetime data, we propose and study a new distribution called the Kumaraswamy Lindley-Poisson (KLP) distribution, which inherits these desirable properties that also cover the shapes of quite a large number of models.

We are also motivated to study the KLP distribution because of the wide and extensive usage of Lindley distribution and the fact that the current generalization still provides a useful means for its continuous extension to more complex situations. An important and positive point of the current generalization is that the Lindley distribution is a basic model or exemplar of the proposed KLP distribution.

This paper is organized as follows. In section 2, the Kumaraswamy-G distribution (see Cordeiro and de Castro[8] for additional details), the model, its sub-models and some statistical properties including expansion of density function, quantile function, hazard function are presented. In section 3, we present the moments. Section 4 contains the distribution of the order statistics and Rényi entropy. Mean deviations, Bonferonni and Lorenz curves are presented in section 5. Maximum likelihood estimates of the model parameters and asymptotic confidence intervals are given in section 6. A simulation study is also presented in section 6. Section 7 contains applications of the proposed model to real data, followed by concluding remarks in section 8.

2. The Model, Sub-models and Properties

The probability density function (pdf) and the corresponding cumulative distribution function (cdf) of the one-parameter Lindley distribution [16] are given by

$$f(x;\beta)=\frac{\beta^2}{\beta+1}(1+x)e^{-\beta x},\quad x>0,\beta>0, \tag{2.1}$$

and

$$F(x)=1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}, \tag{2.2}$$

for $x>0,\alpha,\beta>0$, respectively.

Suppose that the random variable X has the Lindley distribution where its pdf and cdf are given in equations (2.1) and (2.2). Given N, let $X_1,...,X_N$ be independent and identically distributed random variables from Lindley distribution. Let N be distributed according to the zero truncated Poisson distribution [6] with pdf

$$P(N=n)=\frac{\theta^n e^{-\theta}}{n!(1-e^{-\theta})}, \quad n=1,2,...,\theta>0.$$

Let X=max$(Y_1,...,Y_N)$, then the cdf of $X|N=n$ is given by

$$G_{X|N=n}(x) \;=\; \left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]^n, \quad x>0,\ \beta>0,\ \theta>0,$$

which is the exponentiated Lindley distribution. The Lindley-Poisson (LP) distribution denoted by LP(β,θ) is defined by the marginal cdf of X, that is,

$$G_{LP}(x;\beta,\theta) \;=\; \frac{1-\exp\left\{\theta\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]\right\}}{1-e^{\theta}} \tag{2.3}$$

for $x>0,\beta>0,\theta>0$. The LP density function is given by

$$g_{LP}(x;\beta,\theta)=\frac{\theta\beta^2(1+x)e^{-\beta x}\exp\left\{\theta\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]\right\}}{(\beta+1)(e^{\theta}-1)}, \tag{2.4}$$

for $x>0,\beta>0,\theta>0$.

2.1. **Kumaraswamy Lindley Poisson Distribution.** In this sub-section, we present the Kumaraswamy Lindley-Poisson (KLP) distribution and derive some of its properties including the cdf, pdf, expansion of the density, hazard function, quantile function and sub-models.

Consider $G(x)$ to be an arbitrary baseline cdf in the interval $(0,1)$. The cdf $G(x)$, referred to as Kumaraswamy-G distribution [8] has cdf

$$F(x;a,b) \;=\; 1-(1-G(x)^a)^b,$$

where a and b are shape parameters. The pdf of the Kumaraswamy-G distribution is given by

$$f(x;a,b) \;=\; abg(x)[G(x)]^{a-1}[1-G(x)^a]^{b-1}, \quad a>0,b>0, \tag{2.5}$$

where $g(x)=\frac{dG(x)}{dx}$ is the pdf corresponding to the baseline cdf.

By taking $G(x)$ as the cdf of the Lindley-Poisson (LP) distribution in equation (2.5), we obtain the Kumaraswamy Lindley-Poisson (KLP) distribution with a broad class of distributions that may be applicable in a wide range of day to day situations including applications in medicine, reliability and ecology. The cdf of the four-parameter KLP distribution is given by

$$F_{KLP}(x)=1-\left[1-\left(\frac{1-\exp\left\{\theta\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]\right\}}{1-e^{\theta}}\right)^a\right]^b, \tag{2.6}$$

for $x > 0, \theta > 0, \beta > 0, a > 0, b > 0$. The corresponding KLP pdf is given by

$$
\begin{aligned}
f_{KLP}(x) &= \frac{ab\theta\beta^2(1+x)e^{-\beta x}\exp\left\{\theta\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]\right\}}{(\beta+1)(e^{\theta}-1)} \\
&\times \left(\frac{1-\exp\left\{\theta\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]\right\}}{1-e^{\theta}}\right)^{a-1} \\
&\times \left[1-\left(\frac{1-\exp\left\{\theta\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]\right\}}{1-e^{\theta}}\right)^{a}\right]^{b-1} \qquad (2.7)
\end{aligned}
$$

for $x > 0, \beta > 0, \theta > 0, a > 0, b > 0$.

Plots of the pdf of KLP distribution for several values of β, θ, a and b are given in Figures 2.1 and 2.2, respectively. The plots show that the KLP distribution is right skewed and can be decreasing (L shaped).

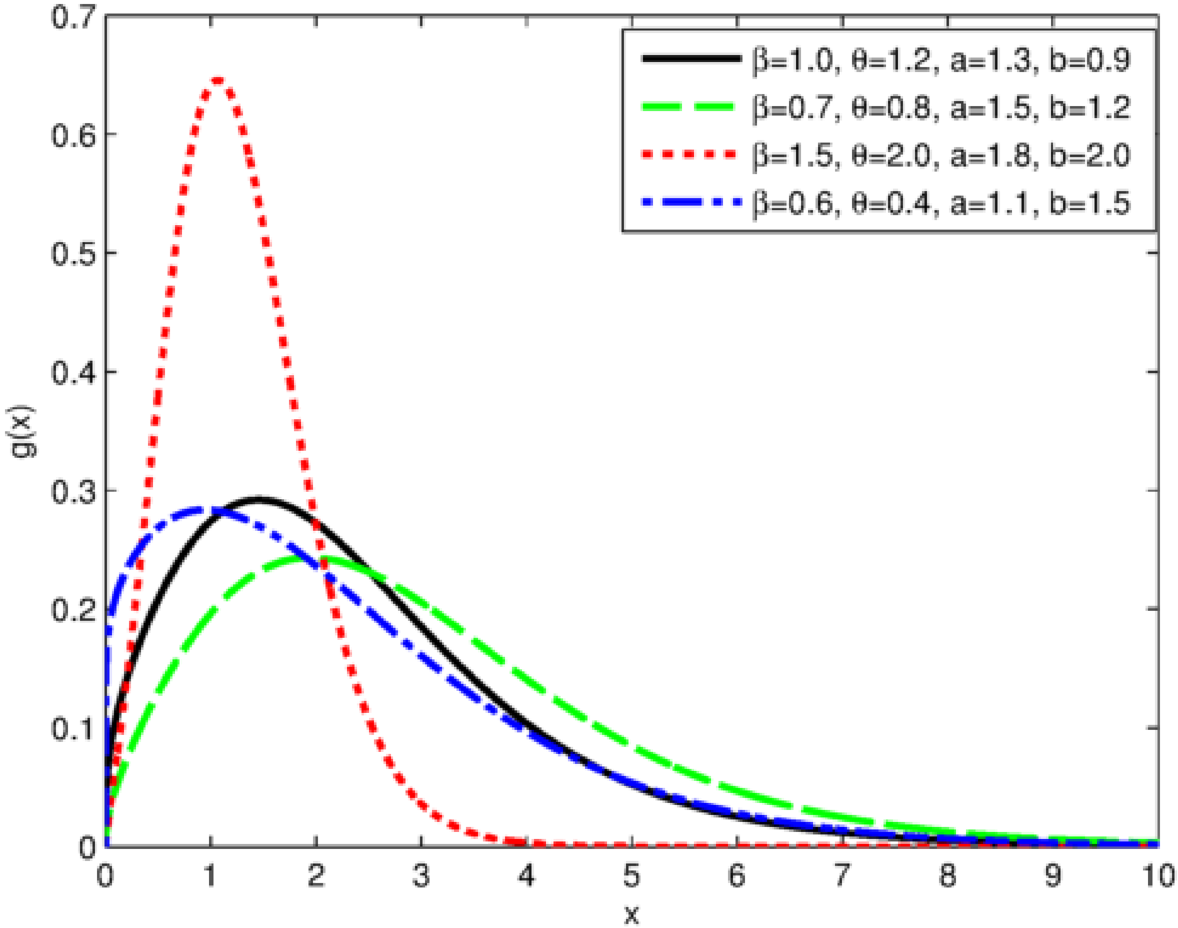

FIGURE 2.1. Plot of the PDF for different values of β, θ, a and b

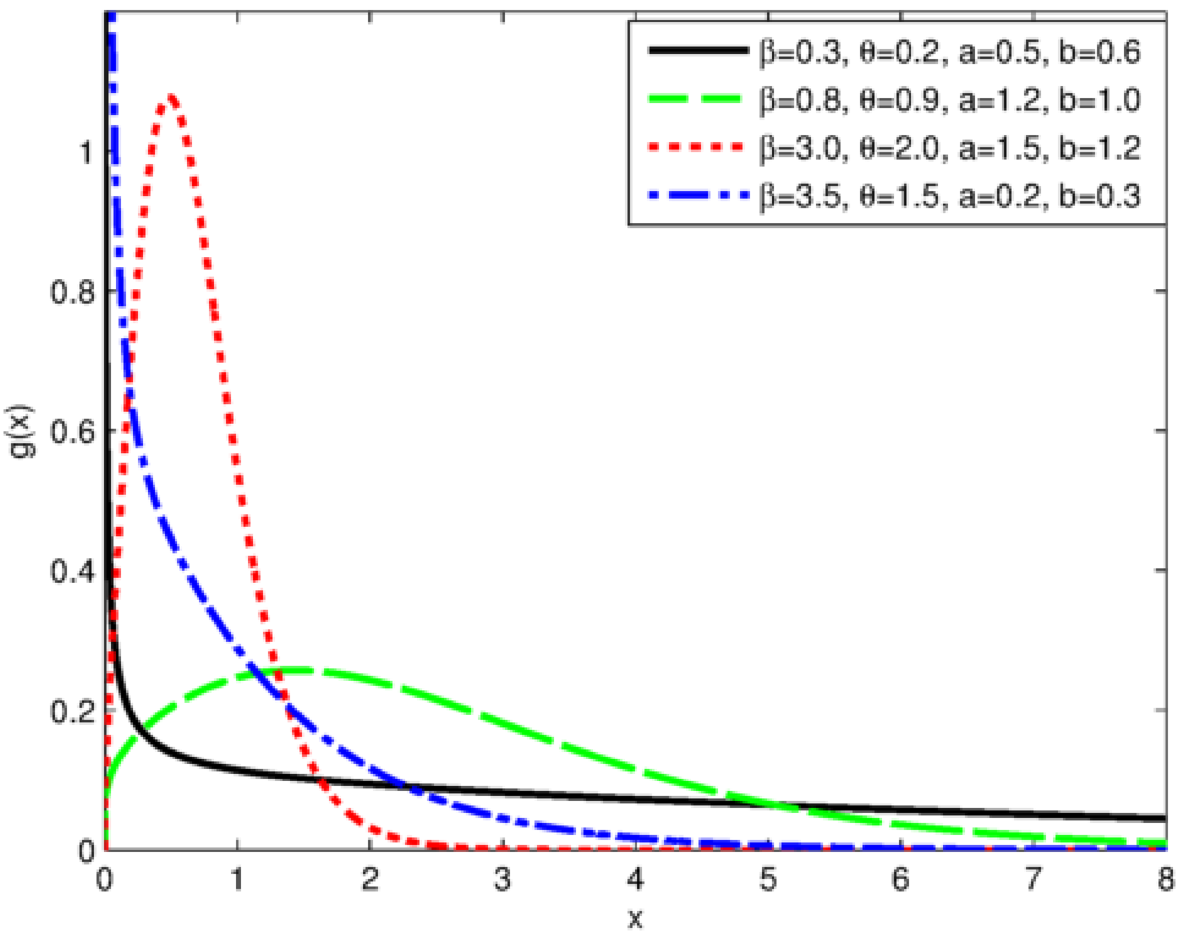

FIGURE 2.2. Plot of the PDF for different values of β, θ, a and b

Using the substitution $\lambda = \lambda(x) = \theta\left[1 - \left(1 + \frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]$, we can write the pdf of the KLP distribution as

$$f_{KLP}(x) = \frac{ab\theta\beta^2(1+x)e^{-\beta x}e^{\lambda}}{(\beta+1)(e^{\theta}-1)}\left(\frac{1-e^{\lambda}}{1-e^{\theta}}\right)^{a-1}\left[1-\left(\frac{1-e^{\lambda}}{1-e^{\theta}}\right)^{a}\right]^{b-1}.$$

2.2. **Expansion of the Density Function.** The expansion of the pdf of KLP distribution is presented in this sub-section. For $b > 0$ a real non-integer, we use the series representation

$$(1 - [G_{LP}(x)]^a)^{b-1} = \sum_{j=0}^{\infty}\binom{b-1}{j}(-1)^j\,[G_{LP}(x)]^{aj},$$

where

$$G_{LP}(x) = G_{LP}(x;\beta,\theta) = \frac{\exp\left\{\theta\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]\right\}-1}{e^{\theta}-1}.$$

We can rewrite the density of the KLP distribution as

$$\begin{aligned}
f_{KLP}(x) &= \sum_{j=0}^{\infty}(-1)^j\binom{b-1}{j}abg_{LP}(x)[G_{LP}(x)]^{aj+a-1} \\
&= \frac{ab\theta\beta^2(1+x)e^{-\beta x}\exp\left\{\theta\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]\right\}}{(\beta+1)(e^{\theta}-1)} \\
&\times \sum_{j=0}^{\infty}(-1)^j\binom{b-1}{j}\left(\frac{1-\exp\left\{\theta\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]\right\}}{1-e^{\theta}}\right)^{aj+a-1} \\
&= \sum_{j,k=0}^{\infty}\frac{(-1)^{aj+a+j+k-1}ab\binom{b-1}{j}\binom{aj+a-1}{k}[e^{\theta(k+1)}-1]}{(e^{\theta}-1)(k+1)} \\
&\times \frac{\theta\beta^2(1+x)e^{-\beta x}\exp\left\{\theta(k+1)\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]\right\}}{(\beta+1)[e^{\theta(k+1)}-1]} \\
&= \sum_{j,k=0}^{\infty}\omega_{j,k}g(x;\beta,\theta(k+1)),
\end{aligned}$$

where

$$\omega_{j,k}=\omega_{j,k}(\theta,a,b)=\frac{(-1)^{aj+a+j+k-1}ab\binom{b-1}{j}\binom{aj+a-1}{k}[e^{\theta(k+1)}-1]}{(e^{\theta}-1)(k+1)} \tag{2.8}$$

and $g(x;\beta,\theta(k+1))$ is the Lindley-Poisson pdf with parameters $\beta>0$ and $\theta(k+1)>0$. This shows that the KLP distribution can be written as a linear combination of Lindley-Poisson density functions. Hence mathematical properties of the KLP distribution can be obtained from those of the LP properties.

2.3. **Survival and Hazard Rate Functions.** The hazard function for the KLP distribution will be presented in this sub-section. Using some selected values of β,θ,a, and b, some plots of the hazard function are presented. The hazard function of the KLP are given respectively by

$$\begin{aligned}
h_{KLP}(x) &= \frac{f_{KLP}(x;\beta,\theta,a,b)}{1-F_{KLP}(x;\beta,\theta,a,b)} \\
&= \frac{ab\theta\beta^2(1+x)e^{-\beta x}e^{\lambda}}{(\beta+1)(e^{\theta}-1)}\left(\frac{1-e^{\lambda}}{1-e^{\theta}}\right)^{a-1}\left[1-\left(\frac{1-e^{\lambda}}{1-e^{\theta}}\right)^{a}\right]^{-1}
\end{aligned}$$

for $x>0,\beta>0,\theta>0,a>0$ and $b>0$, where $\lambda=\theta\left[1-\left(1+\dfrac{\beta x}{\beta+1}\right)e^{-\beta x}\right]$. The graph of the hazard function for various values of the parameters β,θ,a and b are given in Figures 2.3 and 2.4, respectively. These graphs show that the KLP distribution is suitable for monotonic and non-monotonic hazard behaviors which are more likely to be encountered in real life situations.

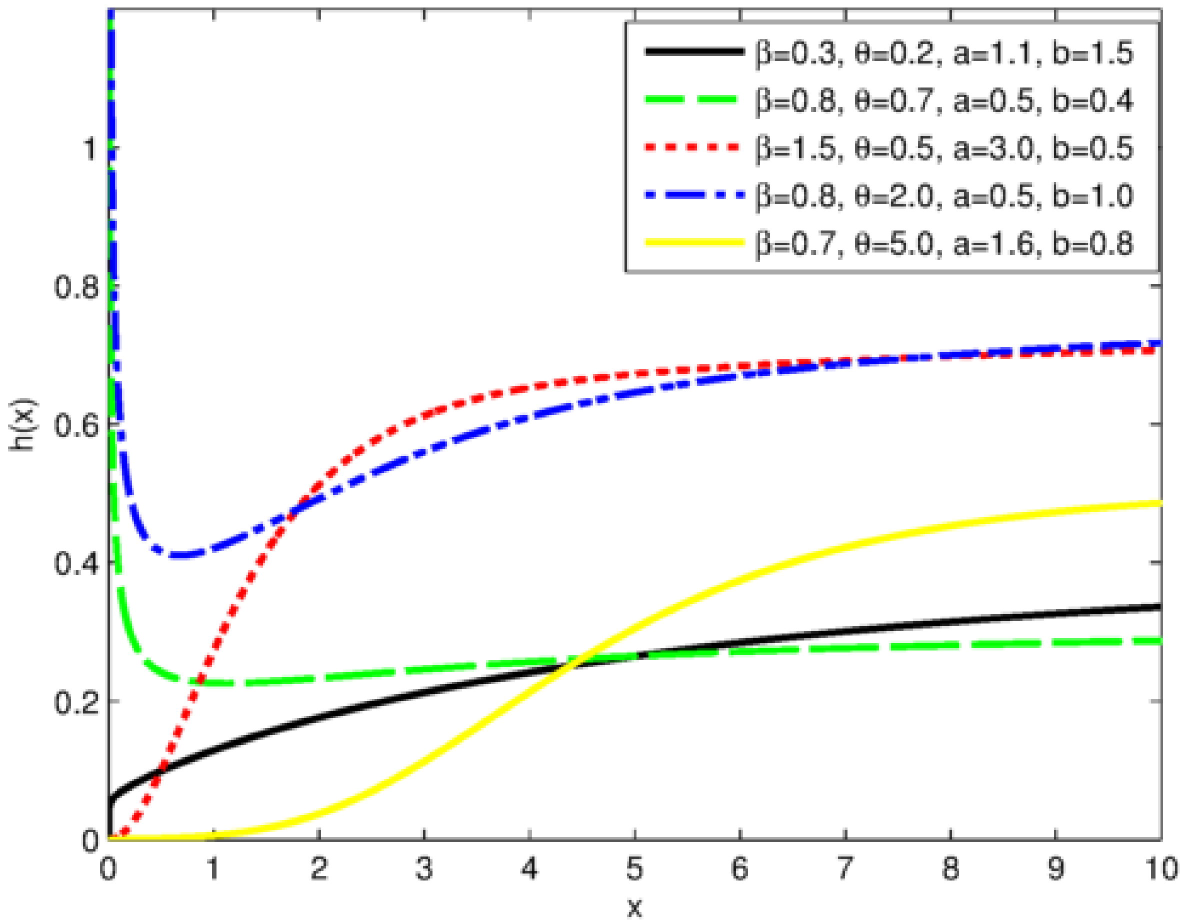

FIGURE 2.3. Plot of the Hazard Function for different values of β, θ, a and b

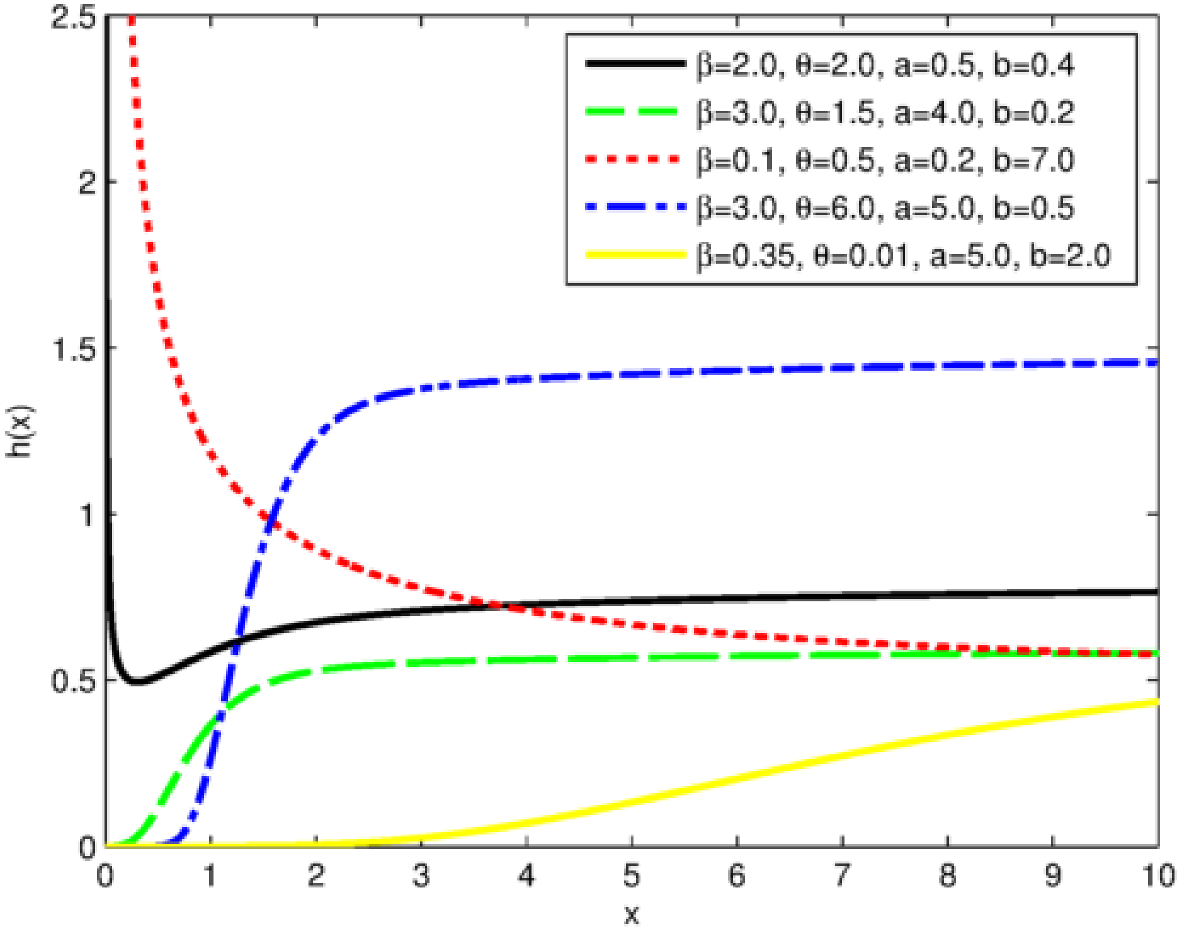

FIGURE 2.4. Plot of the Hazard Function for different values of β, θ, a and b

The graph of the hazard function for different values of the parameters exhibits various shapes such as monotonically increasing and bathtub shapes.

2.4. **Some sub-models of the KLP distribution.** In this sub-section, some sub-models of the KLP distribution for selected values of the parameters β, θ, a and b are presented.

- When $a = b = 1$, we obtain the Lindley Poisson (LP) distribution whose cdf and pdf are given in equations (2.3) and (2.4).
- When $b = 1$, we obtain the exponentiated Lindley Poisson (ELP) distribution which belongs to the resilience parameter family.
- When $a = b = 1$ and $\theta \to 0^+$, the Lindley, (L) distribution becomes a limiting form of the KLP distribution.
- When $b = 1$ and $\theta \to 0^+$, we get the exponentiated Lindley (EL) distribution.
- When $a = 1$, we get the new lifetime distribution belonging to the frailty parameter family

2.5. **Quantile Function.** The quantile function of the KLP distribution is obtained by solving the equation $F(Q(p)) = p$, where $0 < p < 1$. We therefore have

$$1 - \left[1 - \left(\frac{1-\exp\left\{\theta\left[1-\left(1+\frac{\beta Q(p)}{\beta+1}\right)e^{-\beta Q(p)}\right]\right\}}{1-e^{\theta}}\right)^{a}\right]^{b} = p.$$

Using $Z(p) = -1 - \beta - \beta Q(p)$, we have

$$1 - \left[1 - \left(\frac{1-\exp\left\{\theta\left[1+\left(\frac{Z(p)}{\beta+1}\right)\exp(Z(p)+1+\beta)\right]\right\}}{1-e^{\theta}}\right)^{a}\right]^{b} = p,$$

so that

$$Z(p)\exp\{Z(p)\} = \frac{-(\beta+1)}{\exp(1+\beta)}\left\{1 - \frac{1}{\theta}\ln\left[1-(1-e^{\theta})(1-(1-p)^{1/b})^{1/a}\right]\right\},$$

thus,

$$Z(p) = W\left(\frac{-(\beta+1)}{\exp(1+\beta)}\left\{1 - \frac{1}{\theta}\ln\left[1-(1-e^{\theta})(1-(1-p)^{1/b})^{1/a}\right]\right\}\right).$$

for $0 < p < 1$, where $W(.)$ is the Lambert W function [9]. The quantile function of the KLP distribution is obtained by solving for $Q(p)$ in the above equation to obtain

$$Q(p) = -1 - \frac{1}{\beta} - \frac{1}{\beta}W\left(\frac{-(\beta+1)}{\exp(1+\beta)}\left\{1 - \frac{1}{\theta}\ln\left[1-(1-e^{\theta})(1-(1-p)^{1/b})^{1/a}\right]\right\}\right). \quad (2.9)$$

3. Moments

In this section, we present the moments of the KLP distribution. Moments are necessary and important in any statistical analysis, especially in applications. They can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis).

The r^{th} moment of a random variable X following the KLP distribution, denoted by μ'_r is

$$\begin{aligned}\mu'_r &= E(X^r)\\ &= \sum_{j,k=0}^{\infty}\frac{\omega_{j,k}\theta\beta^2}{(\beta+1)[e^{\theta(k+1)}-1]}\int_0^{\infty}x^r(1+x)e^{-\beta x}\\ &\times \exp\left\{\theta(k+1)\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]\right\}dx,\end{aligned} \quad (3.1)$$

where $\omega_{j,k}$ is defined in equation (2.8). In order to find the moments, consider the following lemma:

Lemma 1. Let

$$L_1(\beta,\theta(k+1),a,b,r)=\int_0^\infty x^r(1+x)e^{-\beta x}\exp\left\{\theta(k+1)\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]\right\}dx,$$

then

$$\begin{aligned} L_1(\beta,\theta(k+1),a,b,r) &= \sum_{m=0}^{\infty}\sum_{p=0}^{m}\sum_{q=0}^{p}\sum_{s=0}^{q+1}\binom{m}{p}\binom{p}{q}\binom{q+1}{s}\frac{(-1)^p\theta^m(k+1)^m\beta^q}{m!(\beta+1)^p} \\ &\times \frac{\Gamma(r+s+1)}{[\beta(p+1)]^{r+s+1}}. \end{aligned}$$

Proof. Using the series expansion, $e^z=\sum\limits_{p=0}^{\infty}\frac{z^p}{p!}$, we can rewrite the above integral as

$$\begin{aligned} L_1(\beta,\theta(k+1),a,b,r) &= \sum_{m=0}^{\infty}\frac{\theta^m(k+1)^m}{m!}\int_0^\infty x^r(1+x)e^{-\beta x}\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]^m dx \\ &= \sum_{m=0}^{\infty}\sum_{p=0}^{m}\sum_{q=0}^{p}\sum_{s=0}^{q+1}\binom{m}{p}\binom{p}{q}\binom{q+1}{s}\frac{(-1)^p\theta^m(k+1)^m\beta^q}{m!(\beta+1)^p} \\ &\times \int_0^\infty x^{r+s}e^{-\beta(p+1)x}dx. \end{aligned}$$

By letting $u=\beta(p+1)x$, we have $x=\frac{u}{\beta(p+1)}$ and $dx=\frac{du}{\beta(p+1)}$. Thus

$$\int_0^\infty x^{r+s}e^{-\beta x(p+1)}dx = \frac{\Gamma(r+s+1)}{[\beta(p+1)]^{r+s+1}}.$$

□

By using Lemma 1, the r^{th} moment of the KLP distribution is

$$\mu_r'=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\theta\beta^2\omega_{j,k}}{(\beta+1)[e^{\theta(k+1)}-1]}L_1(\beta,\theta(k+1),a,b,r),$$

4. Order Statistics and Rényi Entropy

In this section, the distribution of order statistics and Rényi entropy for the KLP distribution are presented. The concept of entropy plays a vital role in information theory. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty.

4.1. **Order Statistics and Entropy.** Suppose that $X_1, \cdots, X_n$ is a random sample of size n from a continuous pdf, $f(x)$. Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the corresponding order statistics. If $X_1, \cdots, X_n$ is a random sample from KLP distribution, it follows from equations (2.6) and (2.7) that the pdf of the k^{th} order statistic, say $Y_k = X_{k:n}$ is given by

$$\begin{aligned}
f_k(y_k) &= \frac{n! f_{KLP}(x)}{(k-1)!(n-k)!}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j[F_{KLP}(x)]^{j+k-1} \\
&= \frac{n!}{(k-1)!(n-k)!}\sum_{j=0}^{n-k}\sum_{m=0}^{j+k-1}\sum_{p,q=0}^{\infty}\binom{n-k}{j}\binom{j+k-1}{m}\binom{bm+b-1}{p} \\
&\times \binom{ap+a-1}{q}\frac{(-1)^{j+m+p+q+ap+a-1}ab[e^{\theta(q+1)}-1]}{(e^{\theta}-1)^{ap+p}(q+1)} \\
&\times \frac{\theta(q+1)\beta^2(1+x)e^{-\beta x}e^{\lambda(q+1)}}{(\beta+1)[e^{\theta(q+1)}-1]} \\
&= \sum_{j,m,p,q=0}^{\infty}\varphi_{j,m,p,q}(\beta,\theta,a,b)g(x;\beta,\theta(q+1)),
\end{aligned}$$

where

$$\begin{aligned}
\varphi_{j,m,p,q}(\beta,\theta,a,b) &= \frac{n!}{(k-1)!(n-k)!}\sum_{j,m,p=0}^{\infty}\sum_{q=0}^{\infty}\binom{n-k}{j}\binom{j+k-1}{m}\binom{bm+b-1}{p} \\
&\times \binom{ap+a-1}{q}\frac{(-1)^{j+m+p+q+ap+a-1}ab[e^{\theta(q+1)}-1]}{(e^{\theta}-1)^{ap+p}(q+1)},
\end{aligned}$$

and $g(x;\beta,\theta(q+1))$ is the LP pdf with parameters $\beta > 0$ and $\theta(q+1)) > 0$. Thus, the distribution of the k^{th} order statistic is a linear combination of the LP distribution.

4.2. **Rényi Entropy.** Rényi entropy [24][25] is an extension of Shannon entropy [29][30]. Rényi entropy is defined to be $H_v(f_{KLP}(x;\beta,\theta,a,b)) = \frac{\log\left(\int_0^\infty f_{KLP}^v(x;\beta,\theta,a,b)dx\right)}{1-v}$, where $v > 0$, and $v \neq 1$. Rényi entropy tends to Shannon entropy as $v \to 1$. We therefore have

$$\begin{aligned}
H_v(f_{KLP}) &= \frac{1}{1-v}\left[\log\left(\int_0^\infty f_{KLP}^v(x)dx\right)\right] \\
&= \sum_{i,j,k=0}^{\infty}\sum_{m=0}^{k}\sum_{p=0}^{m}\sum_{q=0}^{p+v}\binom{bv-v}{i}\binom{av+ai-v}{j}\binom{k}{m}\binom{m}{p}\binom{p+v}{q} \\
&\times \frac{(-1)^{i+av+ai-v+j+m}[\theta(j+v)]^k\beta^p}{(\beta+1)^m k!(e^\theta-1)^{av+ai-v}}\int_0^\infty x^q e^{-\beta(v+k)x}dx\Bigg] \\
&= \frac{1}{1-v}\log\Bigg[\sum_{i,j,k=0}^{\infty}\sum_{m=0}^{k}\sum_{p=0}^{m}\sum_{q=0}^{p+v}\binom{bv-v}{i}\binom{av+ai-v}{j} \\
&\times \binom{k}{m}\binom{m}{p}\binom{p+v}{q}\frac{(-1)^{v-av+k+p}[\theta(v+k]^m\beta^q}{(e^\theta-1)^{j+av-v}(\beta+1)^p m!} \\
&\times \left(\frac{ab\theta\beta^2}{(\beta+1)(e^\theta-1)}\right)^v\frac{\Gamma(q+1)}{[\beta(v+k)]^{q+1}}\Bigg],
\end{aligned}$$

for $v>0, v\neq 1$.

5. Mean Deviations, Bonferroni and Lorenz Curves

Deviations from the mean and median help in giving a sense of the amount of spread in a population. The mean deviation about the mean and mean deviation about the median of the KLP distribution are given by

$$D(\mu) = 2\mu F_{KPL}(\mu) - 2\mu + \sum_{j,k=0}^{\infty}\frac{\theta\beta^2\omega_{j,k}}{(\beta+1)[e^{\theta(k+1)}-1]}L_2(\beta,\theta(k+1),a,b,1,\mu)$$

and

$$D(M) = -\mu + \sum_{j,k=0}^{\infty}\frac{\theta\beta^2\omega_{j,k}}{(\beta+1)[e^{\theta(k+1)}-1]}L_2(\beta,\theta(k+1),a,b,1,M).$$

Consequently, Lorenz and Bonferroni curves are given by

$$L(F_{KLP}(x)) = \frac{\int_0^y tf_{KLP}(t)dt}{E(X)}, \quad \text{and} \quad B(F_{KLP}(x)) = \frac{L(F_{KLP}(x))}{F_{KLP}(x)},$$

or

$$L(p) = \frac{1}{\mu}\int_0^q tf_{KLP}(t)dt, \quad \text{and} \quad B(p) = \frac{1}{p\mu}\int_0^q tf_{KLP}(t)dt,$$

respectively, where $q = F_{KLP}^{-1}(p)$.

6. Maximum Likelihood Estimation

Let $x_1, \cdots, x_n$ be a random sample from the KLP distribution. The log-likelihood function is given by

$$\begin{aligned} L &= n\log(a) + n\log(b) + n\log(\theta) + 2n\log(\beta) - n\log(\beta+1) \\ &+ \sum_{i=1}^{n}\log(1+x_i) - \beta\sum_{i=0}^{n}x_i + \sum_{i=1}^{n}\lambda_i + (a-1)\sum_{i=1}^{n}\log(e^{\lambda_i}-1) \\ &+ (b-1)\sum_{i=1}^{n}\log\left[(e^{\theta}-1)^a - (e^{\lambda_i}-1)^a\right] - nab\log(e^{\theta}-1). \end{aligned}$$

The elements of the score vector are given by

$$\begin{aligned} \frac{\partial L}{\partial a} &= \frac{n}{a} + \sum_{i=1}^{n}\log(e^{\lambda_i}-1) - nb\log(e^{\theta}-1) \\ &+ (b-1)\sum_{i=1}^{n}\frac{(e^{\theta}-1)^a\log(e^{\theta}-1) - (e^{\lambda_i}-1)^a\log(e^{\lambda_i}-1)}{(e^{\theta}-1)^a - (e^{\lambda_i}-1)^a}, \end{aligned}$$

$$\frac{\partial L}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\log\left[(e^{\theta}-1)^a - (e^{\lambda_i}-1)^a\right] - na\log(e^{\theta}-1),$$

$$\begin{aligned} \frac{\partial L}{\partial \beta} &= \frac{2n}{\beta} - \frac{n}{\beta+1} - \sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\frac{\partial\lambda_i}{\partial\beta} + (a-1)\sum_{i=1}^{n}\frac{\frac{\partial\lambda_i}{\partial\beta}e^{\lambda_i}}{e^{\lambda_i}-1} \\ &- a(b-1)\sum_{i=1}^{n}\frac{e^{\lambda_i}(e^{\lambda_i}-1)^{a-1}\frac{\partial\lambda_i}{\partial\beta}}{(e^{\theta}-1)^a - (e^{\lambda_i}-1)^a} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= \frac{n}{\theta} + \sum_{i=1}^{n}V(x_i) + \sum_{i=1}^{n}\frac{V(x_i)e^{\lambda_i}}{e^{\lambda_i}-1} - \frac{nabe^{\theta}}{e^{\theta}-1} \\ &+ a(b-1)\sum_{i=1}^{n}\frac{e^{\theta}(e^{\theta}-1)^{a-1} - V(x_i)e^{\lambda_i}(e^{\lambda_i}-1)^{a-1}}{(e^{\theta}-1)^a - (e^{\lambda_i}-1)^a}, \end{aligned}$$

respectively. Note that since $\lambda = \theta\left[1 - \left(1 + \frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]$, we have

$$\frac{\partial\lambda}{\partial\beta} = \theta e^{-\beta x}\left[\left(1 + \frac{\beta x}{\beta+1}\right) - \frac{1}{(\beta+1)^2}\right]$$

and

$$\frac{\partial\lambda}{\partial\theta} = \left[1 - \left(1 + \frac{\beta x}{\beta+1}\right)e^{-\beta x}\right] = V(x).$$

The maximum likelihood estimates, $\hat{\Delta}$ of $\Delta = (\beta, \theta, a, b)$ are obtained by solving the nonlinear equations $\frac{\partial l}{\partial\beta} = 0$, $\frac{\partial l}{\partial\theta} = 0$, $\frac{\partial l}{\partial a} = 0$, and $\frac{\partial l}{\partial b} = 0$. These equations are not in closed form and the values of the parameters β, θ, a and b must be found by using iterative methods.

We maximize the likelihood function using NLmixed procedure in SAS as well as the function nlm in R ([32]). These functions were applied and executed for a wide range of initial values. This process often results or leads to more than one maximum, however, in these cases, we take the MLEs corresponding to the largest value of the maxima. In a few cases, no maximum was identified for the selected initial values. In these cases, a new initial value was tried in order to obtain a maximum.

The issues of existence and uniqueness of the MLEs are of theoretical interest and have been studied by several authors for different distributions including [28], [27], [34], and [33]. At this point we are not able to address the theoretical aspects (existence, uniqueness) of the MLE of the parameters of the KLP distribution.

Note that the KLP density $f_{KLP}(\cdot;\mathbf{\Delta})$ has second derivatives with respect to the parameters, so that Fisher information matrix (FIM), $\mathbf{I}_{ij}(\mathbf{\Delta})$ can be expressed as

$$\mathbf{I}_{ij}(\mathbf{\Delta}) = E_{\mathbf{\Delta}}\left[\frac{\partial^2 \log(f_{KLP}(X;\mathbf{\Delta}))}{\partial\delta_i\partial\delta_j}\right], i,j = 1,2,3,4.$$

Elements of the FIM can be numerically obtained by MATLAB or MAPLE software. The total FIM $\mathbf{I}_n(\mathbf{\Delta})$ can be approximated by

$$\mathbf{J}_n(\hat{\mathbf{\Delta}}) \approx \left[-\left.\frac{\partial^2 \log L}{\partial\delta_i\partial\delta_j}\right|_{\mathbf{\Delta}=\hat{\mathbf{\Delta}}}\right]_{4\times 4}, i,j = 1,2,3,4. \tag{6.1}$$

For real data, the matrix given in equation (6.1) is obtained after the convergence of the Newton-Raphson procedure in MATLAB or R software. Let $\hat{\mathbf{\Delta}} = (\hat{\beta}, \hat{\theta}, \hat{a}, \hat{b})$ be the maximum likelihood estimate of $\mathbf{\Delta} = (\beta, \theta, a, b)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\hat{\mathbf{\Delta}} - \mathbf{\Delta}) \xrightarrow{d} N_4(\underline{\mathbf{0}}, \mathbf{I}^{-1}(\mathbf{\Delta}))$, where $\mathbf{I}(\mathbf{\Delta})$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $\mathbf{I}(\mathbf{\Delta})$ is replaced by the observed information matrix evaluated at $\hat{\mathbf{\Delta}}$, that is $\mathbf{J}(\hat{\mathbf{\Delta}})$. The multivariate normal distribution with mean vector $\underline{0} = (0,0,0,0)^T$ and covariance matrix $\mathbf{I^{-1}}(\mathbf{\Delta})$ can be used to construct confidence intervals for the model parameters. That is, the approximate $100(1-\eta)\%$ two-sided confidence intervals for β, θ, a and b are given by

$$\hat{\beta} \pm Z_{\eta/2}\sqrt{\mathbf{I}^{-1}_{\beta\beta}(\hat{\mathbf{\Delta}})}, \quad \hat{\theta} \pm Z_{\eta/2}\sqrt{\mathbf{I}^{-1}_{\theta\theta}(\hat{\mathbf{\Delta}})}, \quad \hat{a} \pm Z_{\eta/2}\sqrt{\mathbf{I}^{-1}_{aa}(\hat{\mathbf{\Delta}})},$$

and $\hat{b} \pm Z_{\eta/2}\sqrt{\mathbf{I}^{-1}_{bb}(\hat{\mathbf{\Delta}})}$, respectively, where $\mathbf{I}^{-1}_{\beta\beta}(\hat{\mathbf{\Delta}}), \mathbf{I}^{-1}_{\theta\theta}(\hat{\mathbf{\Delta}}), \mathbf{I}^{-1}_{aa}(\hat{\mathbf{\Delta}})$ and $\mathbf{I}^{-1}_{bb}(\hat{\mathbf{\Delta}})$ are diagonal elements of $\mathbf{I}_n^{-1}(\hat{\mathbf{\Delta}}) = (n\mathbf{I}(\hat{\mathbf{\Delta}}))^{-1}$ and $Z_{\eta/2}$ is the upper $(\eta/2)^{th}$ percentile of a standard normal distribution.

We can use the likelihood ratio (LR) test to compare the fit of the KLP distribution with its sub-models for a given data set. For example, to test $a = b = 1$, the LR statistic is $\omega^* = 2[\ln(L(\hat{\beta}, \hat{\theta}, \hat{a}, \hat{b})) - \ln(L(\tilde{\beta}, \tilde{\theta}, 1, 1))]$, where $\hat{\beta}$, $\hat{\theta}$, $\hat{a}$ and $\hat{b}$ are the unrestricted estimates, and $\tilde{\beta}$ and $\tilde{\theta}$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega^* > \chi^2_\epsilon$, where χ^2_ϵ denote the upper $100\epsilon\%$ point of the χ^2 distribution with 2 degrees of freedom.

6.1. **Monte Carlo Simulation Study.** In this sub-section, we study the performance of the maximum likelihood method for estimating the KLP model parameters by conducting simulations for different sample sizes and different parameter values. Equation in (2.9) was used to generate random data from the KLP distribution. The simulation study was repeated $N = 1,000$ times each with samples of size $n = 100, 200, 400, 800, 1000$ and parameter values $I : \beta = 0.5, \theta = 0.4, a = 0.3, b = 0.5$ and $II : \beta = 2.0, \theta = 2.0, a = 0.5, b = 0.5$. Four quantities were computed in this simulation study:

(a) Average bias of the MLE $\hat{\vartheta}$ of the parameter $\vartheta = \beta, \theta, a, b :$

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{\vartheta} - \vartheta).$$

(b) Root mean squared error (RMSE) of the MLE $\hat{\vartheta}$ of the parameter $\vartheta = \beta, \theta, a, b :$

$$\sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\vartheta} - \vartheta)^2}.$$

(c) Coverage probability (CP) of 95% confidence intervals of the parameter $\vartheta = \beta, \theta, a, b$, i.e., the percentage of intervals that contain the true value of the parameter ϑ.

(d) Average width (AW) of 95% confidence intervals of the parameter $\vartheta = \beta, \theta, a, b$.

Table 6.1 presents the Average Bias, RMSE, CP and AW values of the parameters β, θ, a and b for different sample sizes. According to the results, it can be concluded that as the sample size n increases, the RMSEs decay toward zero. We also observe that for all the parameters, the biases decrease as the sample size n increases. Also, the results show that the coverage probabilities of the confidence intervals are quite close to the nominal level of 95% and that the average confidence widths decrease as the sample size increases.

Table 6.1. Monte Carlo Simulation Results: Average Bias, RMSE, CP and AW

		I				II			
Parameter	n	Average Bias	RMSE	CP	AW	Average Bias	RMSE	CP	AW
β	100	-0.00501	0.24462	0.9510	1.51624	-0.10803	0.87209	0.9410	5.20699
	200	-0.00295	0.23610	0.9490	1.13275	-0.03140	0.76234	0.9530	4.25096
	400	-0.00043	0.20765	0.9480	0.91630	-0.02458	0.64276	0.9580	3.24892
	800	-0.00027	0.18282	0.9450	0.70245	-0.01504	0.49704	0.9550	2.38966
	1000	-0.00019	0.16592	0.9550	0.66421	-0.00511	0.45924	0.9580	2.08619
θ	100	0.88197	2.34717	0.9610	9.92480	0.84490	1.69587	0.9520	7.14813
	200	0.65401	1.90481	0.9520	7.38017	0.40583	1.08716	0.9480	4.51975
	400	0.61498	1.35036	0.9490	5.00112	0.18632	0.74348	0.9450	3.01703
	800	0.47330	0.96665	0.9460	3.40629	0.07429	0.49949	0.9520	2.02828
	1000	0.38674	0.79037	0.9510	2.87423	0.05577	0.45991	0.9490	1.80380
a	100	0.00487	0.05577	0.9480	0.29526	-0.00084	0.13179	0.9370	0.59255
	200	0.00189	0.04795	0.9490	0.21880	-0.00056	0.10147	0.9420	0.43481
	400	0.00046	0.03968	0.9530	0.16529	-0.00019	0.08019	0.9450	0.31945
	800	0.00037	0.03462	0.9610	0.12181	-0.00013	0.05631	0.9480	0.22884
	1000	0.00016	0.02995	0.9530	0.11095	-0.00103	0.05113	0.9530	0.20353
b	100	0.89313	1.65094	0.9450	10.26548	0.77219	1.75168	0.9640	10.59793
	200	0.88861	1.54050	0.9490	7.79171	0.48749	1.11044	0.9540	5.20265
	400	0.60264	1.13678	0.9510	4.38831	0.29611	0.72701	0.9480	2.79117
	800	0.42173	0.75993	0.9420	2.50884	0.13845	0.34273	0.9560	1.21017
	1000	0.31041	0.56817	0.9480	1.84970	0.12892	0.30152	0.9510	1.03431

7. Applications

In this section, the KLP distribution is applied to real data sets in order to illustrate the usefulness and applicability of the model. We fit the density functions of the Kumaraswamy Lindley-Poisson (KLP), Lindley-Poisson (LP) and Lindley (L) distributions. For comparison purposes, we also fit the beta exponentiated Lindley (BEL) distribution [22] which is also a 4 parameter model comparable to the KLP distribution. The pdf of the BEL distribution is given by

$$\begin{aligned} f_{BEL}(x;\beta,\theta,a,b) &= \frac{\beta^2\theta}{B(a,b)(\beta+1)}(1+x)e^{-\beta x} \\ &\times \left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]^{\theta a-1} \\ &\times \left\{1-\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]^{\theta}\right\}^{b-1} \end{aligned}$$

for $x>0,\beta>0,\theta>0,a>0,b>0$. Estimates of the parameters of the distributions, standard errors (in parentheses), Akaike Information Criterion ($AIC = 2p - 2\log(\hat{L})$), Consistent Akaike Information Criterion ($AICC = AIC + \frac{2p(p+1)}{n-p-1}$), Bayesian Information Criterion ($BIC = p\log(n) - 2\log(\hat{L})$), where $\hat{L} = L(\hat{\Delta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are obtained.

The first data set consists of 119 observations on fracture toughness of Alumina (Al_2O_3)(in the units of MPa $m^{1/2}$). This data was studied by Nadarajah and Kotz [19]. The data is available at and taken from the following web-site:
http://www.ceramics.nist.gov/.srd/summary/ftmain.htm.

The second data set gives failure times of a sample of $n = 101$ aluminum specimens of type 6061-T6 obtained by Birnbaum and Saunders [4]. These specimens were cut parallel to the direction of rolling and oscillating at 18 cycles per seconds and they were exposed to a pressure with maximum stress of 31,000 pounds per square inch (psi). The specimens were tested until failure.

The third data set that is fitted to the KLP distribution consists of breaking stress of carbon fibers which was analyzed by Bader and Priest [2]. The data represent the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20 mm.

The fourth data set set consists of 63 observations of the strengths of 1.5 cm glass fibres, originally obtained by workers at the UK National Physical Laboratory. The data was also studied by Smith and Naylor [31].

Table 7.1 displays results obtained from analyzing the Silicone Nitride data studied by Nadarajah and Kotz [19]. Estimates of the parameters of KLP distribution (s-tandard error in parentheses), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Kolmogorov-Smirnov (K-S) statistic and its p-value are given in the Table 7.1. Plots of the fitted densities and histogram, and observed probability versus predicted probability for the Silicone Nitride data are given in Figures 7.1 and 7.2, respectively. For the probability plot, we plotted $F_{KLP}(x_{(j)}; \hat{\beta}, \hat{\theta}, \hat{a}, \hat{b})$ against $\dfrac{j-0.375}{n+0.25}$, $j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares

$$SS = \sum_{j=1}^{n}\left[F_{KLP}(x_{(j)}) - \left(\frac{j-0.375}{n+0.25}\right)\right]^2.$$

Table 7.1. Estimates of Models for Silicone Nitride Data

Model	Estimates β	θ	a	b	Statistics $-2\log L$	AIC	$AICC$	BIC	SS	K-S	p-value
KLP(β,θ,a,b)	0.3310	4.3514	2.2709	108.36	336.9	344.9	345.3	356.0	0.0651	0.063963	0.7150
	(0.2214)	(5.9017)	(2.2192)	(339.92)							
LP$(\beta,\theta,1,1)$	1.1076	22.9292	1	1	368.3	372.3	372.4	377.8	0.5760	0.142930	0.01547
	(0.0623)	(4.7641)									
L$(\beta,-,1,1)$	0.3967	-	1	1	534.8	536.8	536.8	539.6	4.9662	0.335270	4.812×10^{-12}
	(0.0263)										
BEL(β,θ,a,b)	0.2120	6.0217	0.6111	431.95	337.0	345.0	345.4	356.2	0.0784	0.070035	0.6037
	(0.1067)	(4.6244)	(0.4978)	(49.4370)							

The LR test statistics of the hypotheses H_0 : LP against H_a : KLP, H_0 : L against H_a : KLP, and H_0 : L against H_a :LP are 27.4 (p-value=$1.12\times10^{-6} < 0.001$), 197.9 (p-value=$1.20\times10^{-42} < 0.001$) and 166.5 (p-value=$4.30\times10^{-38} < 0.001$). We can therefore conclude that there is a significant difference between KLP and LP distributions, KLP and L distributions as well as between LP and L distributions. The values of the statistics AIC, AICC and BIC are very close for the KLP and the BEL distributions. However, the KLP distribution has the smallest K-S statistic and the largest p-value indicating that KLP distribution provides a better fit among the four models when fitting the Silicone Nitride data. The KLP yields the smallest value for Sum of Squares (SS) among all the models.

Table 7.2. Estimates of Models for Aluminum Specimens Data

	Estimates				Statistics						
Model	β	θ	a	b	$-2\log L$	AIC	$AICC$	BIC	SS	K-S	p-value
KLP(β,θ,a,b)	0.0313	6.1948	3.9159	4.3329	904.4	912.4	912.8	922.8	0.0413	0.061896	0.8382
	(0.0062)	(23.8570)	(14.5997)	(2.4939)							
LP$(\beta,\theta,1,1)$	0.0532	96.3691	1	1	914.7	918.7	918.8	923.9	0.1759	0.100170	0.2682
	(0.0033)	(32.6069)									
L$(\beta,-,1,1)$	0.0148	-	1	1	1106.3	1108.3	1108.3	1110.9	4.8973	0.400330	2.398×10^{-14}
	(0.0011)										
BEL(β,θ,a,b)	0.0318	31.6377	0.6532	4.2752	904.9	912.9	912.3	923.3	0.0465	0.067242	0.7565
	(0.0068)	(98.0430)	(2.2797)	(6.0065)							

Table 7.2 gives the estimates of the model parameters and the statistics AIC, AICC, BIC, K-S statistic and its p-value for the Aluminum specimen data. The LR test statistics of the hypotheses H_0 : LP against H_a : KLP, H_0 : L against H_a : KLP, and H_0 : L against H_a :LP are 10.3 (p-value=$5.8\times10^{-3} < 0.01$), 201.9 (p-value=$1.64\times10^{-43} < 0.001$) and 191.6 (p-value=$1.42\times10^{-43} < 0.001$). We can therefore conclude that there is a significant difference between KLP and LP distributions, KLP and L distributions as well as between LP and L distributions. The values of the statistics AIC, AICC and BIC are very close for the KLP and the BEL distributions. However, the KLP distribution has the smallest K-S statistic and the largest p-value indicating that KLP distribution provides a better fit among the four models when fitting the Aluminum Specimens Data. The KLP yields the smallest value for Sum of Squares (SS) among all the models. Plots of the fitted densities and histogram, observed probability versus predicted probability for the Aluminum specimen data are given in Figures 7.3 and 7.4. The plot of the fitted pdf support the conclusion based on Table 6.2. The figures suggest that both KLP and BEL distributions captures the middle part of the data, as well as the tails better than the fitted sub-models.

Table 7.3 shows the results obtained from analyzing the carbon fibers data of Bader and Priest [2].

Table 7.3. Estimates of Models for Carbon Fibers Data

	Estimates				Statistics						
Model	β	θ	a	b	$-2\log L$	AIC	$AICC$	BIC	SS	K-S	p-value
KLP(β,θ,a,b)	0.7230	0.0348	8.3910	24.7633	97.7	105.7	106.3	114.7	0.0136	0.038759	0.9999
	(6.2713)	(37.1709)	(33.3868)	(429.21)							
LP$(\beta,\theta,1,1)$	2.3266	67.1553	1	1	107.8	111.8	112.0	116.3	0.1332	0.088736	0.6489
	(0.1808)	(24.3584)									
L$(\beta,-,1,1)$	0.6545	-	1	1	238.4	240.4	240.4	242.6	3.4678	0.401130	4.547×10^{-10}
	(0.05803)										
BEL(β,θ,a,b)	1.3696	48.7520	0.3328	7.2260	97.9	105.9	106.5	114.8	0.0161	0.041807	0.9997
	(0.4863)	(99.3139)	(0.5132)	(2.0853)							

The LR test statistic of the hypothesis H_0 : LP against H_a : KLP, H_0 : L against H_a : KLP, and H_0 : L against H_a :LP are 10.1 (p-value=6.4×10^{-3}, 140.7 (p-value=$2.67\times10^{-30} < 0.001$) and 130.6 (p-value=$3.03\times10^{-30} < 0.001$). We can therefore conclude that there is a significant difference between KLP and LP distributions, KLP and L distributions as well as between LP and L distributions. The values of the statistics AIC, AICC and BIC are very close for the KLP and the BEL distributions. However, the KLP distribution has the smallest K-S statistic and the largest p-value indicatiing that KLP distribution provides a better fit among the four models when fitting the Carbon Fiber data. The KLP yields the smallest value for Sum of Squares (SS) among all the models.

Plots of the fitted densities and histogram, observed probability versus predicted probability for the Carbon Fiber data are given in Figures 7.5 and 7.6, respectively. The plot of the fitted pdf support the conclusion based on Table 7.4. The figures suggest that both KLP and BEL distributions capture the middle part of the data, as well as the tails better than the fitted sub-models.

Table 7.4. Estimates of Models for Glass Fibers Data

	Estimates				Statistics						
Model	β	θ	a	b	$-2\log L$	AIC	$AICC$	BIC	SS	K-S	p-value
KLP(β,θ,a,b)	0.7334	11.8621	1.2658	2083.13	28.2	36.2	36.9	44.7	0.1529	0.13306	0.2146
	(0.3463)	(22.4170)	(2.3915)	(9883.45)							
LP$(\beta,\theta,1,1)$	3.0456	29.3432	1	1	59.3	63.3	63.5	67.6	0.7157	0.21887	0.004783
	(0.2338)	(8.3871)									
L$(\beta,-,1,1)$	0.9961	-	1	1	162.6	164.6	164.6	166.7	3.3017	0.38642	1.349×10^{-8}
	(0.0948)										
BEL(β,θ,a,b)	0.4705	8.0340	0.5670	8989.04	29.2	37.2	37.8	45.7	0.1951	0.14755	0.1287
	(0.1995)	(4.2325)	(0.3193)	(124.49)							

Plots of the fitted densities and histogram and probability plots for the glass fibres data from Smith and Naylor [31] are given in Figures 7.7 and 7.8 respectively. The LR test statistic for the test of the hypotheses H_0 : LP against H_a : KLP, H_0 : L against H_a : KLP, and H_0 : L against H_a :LP are 31.1 (p-value=1.765×10^{-7}, 134.4 (p-value=$6.092\times10^{-29} < 0.0001$) and 103.3 (p-value=$2.88\times10^{-24} < 0.0001$), respectively. We can therefore conclude that there is a significant difference between KLP and LP distributions, KLP and L distributions as well as between LP and L distributions.

The values of AIC, AICC and BIC shows that the KLP distribution is a better model and the SS value is comparatively smaller than the corresponding values for the LP and L distributions. The values of these statistics (AIC, AICC, BIC) for the KLP distribution are very competitive when compared to those of the non-nested BEL distribution. However, the KLP distribution has the smallest K-S statistic and the largest p-value indicating that KLP distribution provides a better fit among the four models when fitting the Glass Fiber data. The plot of the fitted pdf support the conclusion based on Table 7.4. The figures suggest that both KLP and BEL distributions captures the middle part of the data, as well as the tails better than the fitted sub-models.

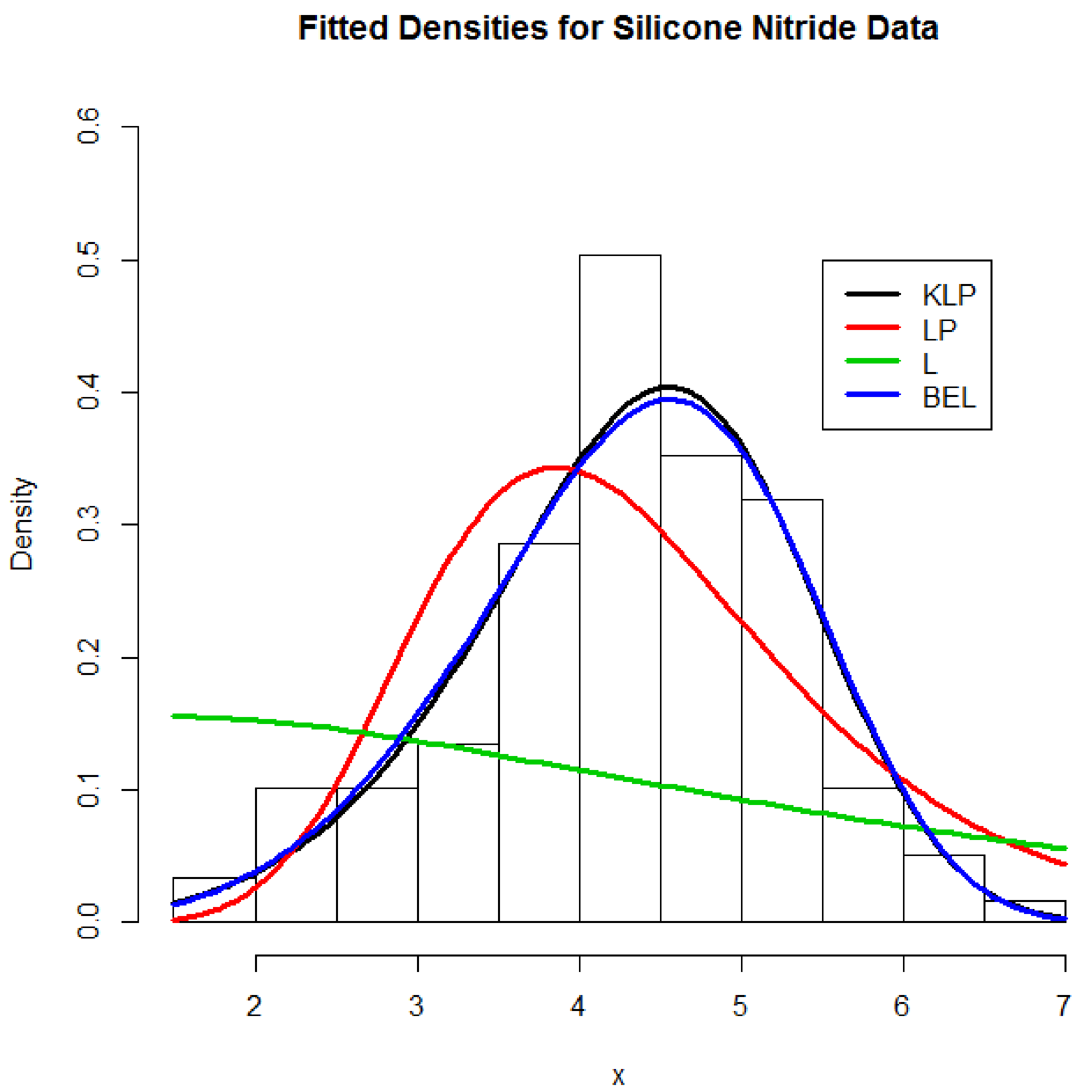

FIGURE 7.1. Histogram and Fitted Density for Silicone Nitride Data

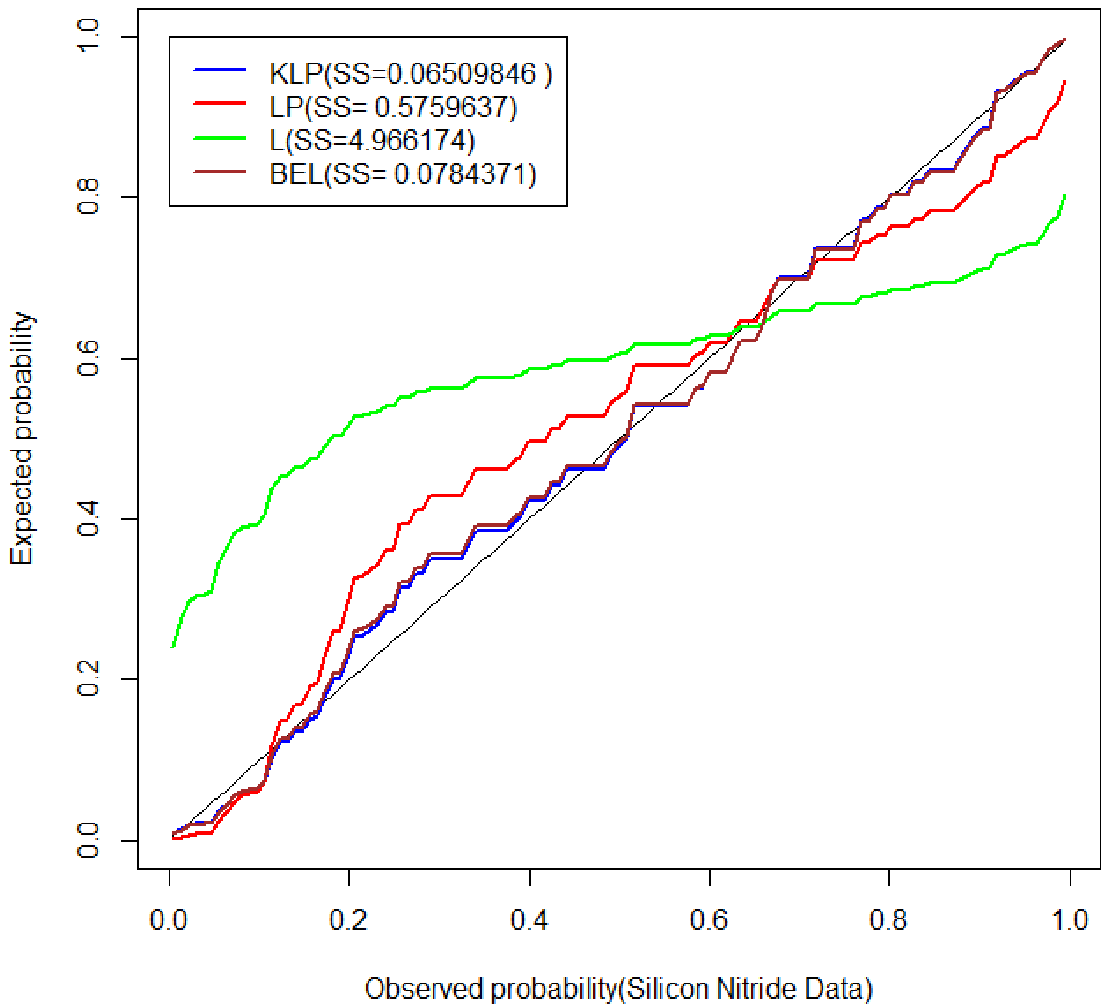

FIGURE 7.2. Probability Plots for Silicone Nitride Data

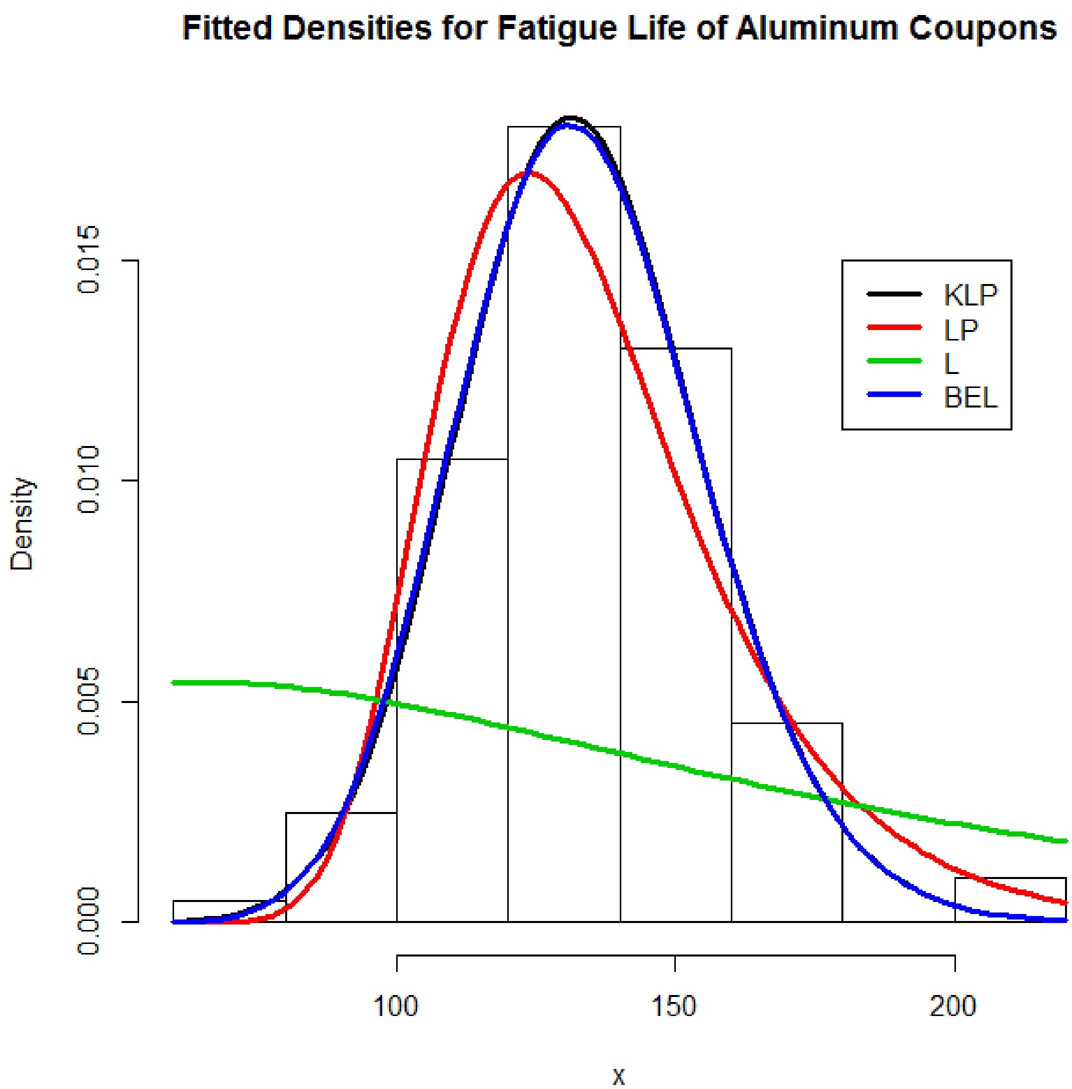

FIGURE 7.3. Histogram and Fitted Density for Aluminum Specimens Data

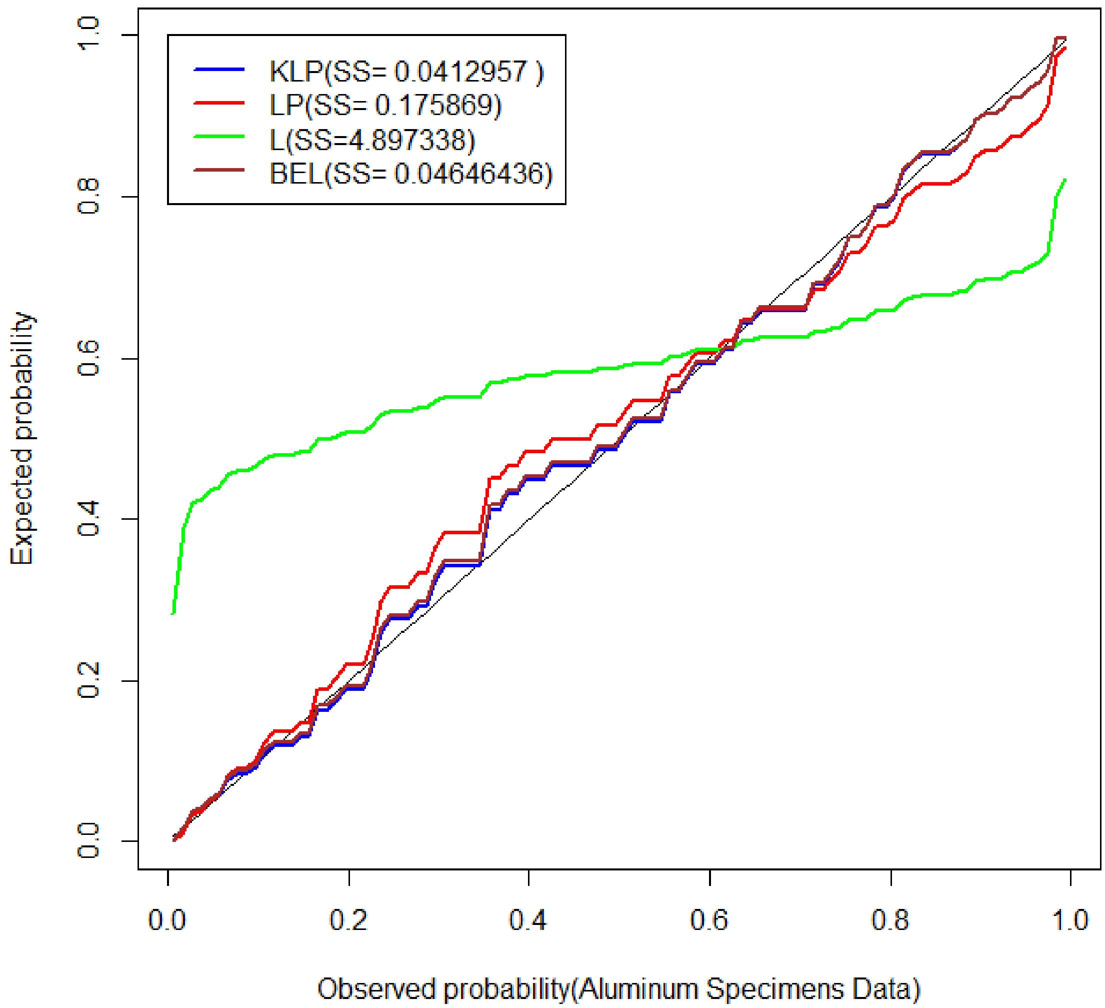

Figure 7.4. Probability Plots for Aluminum Specimens Data

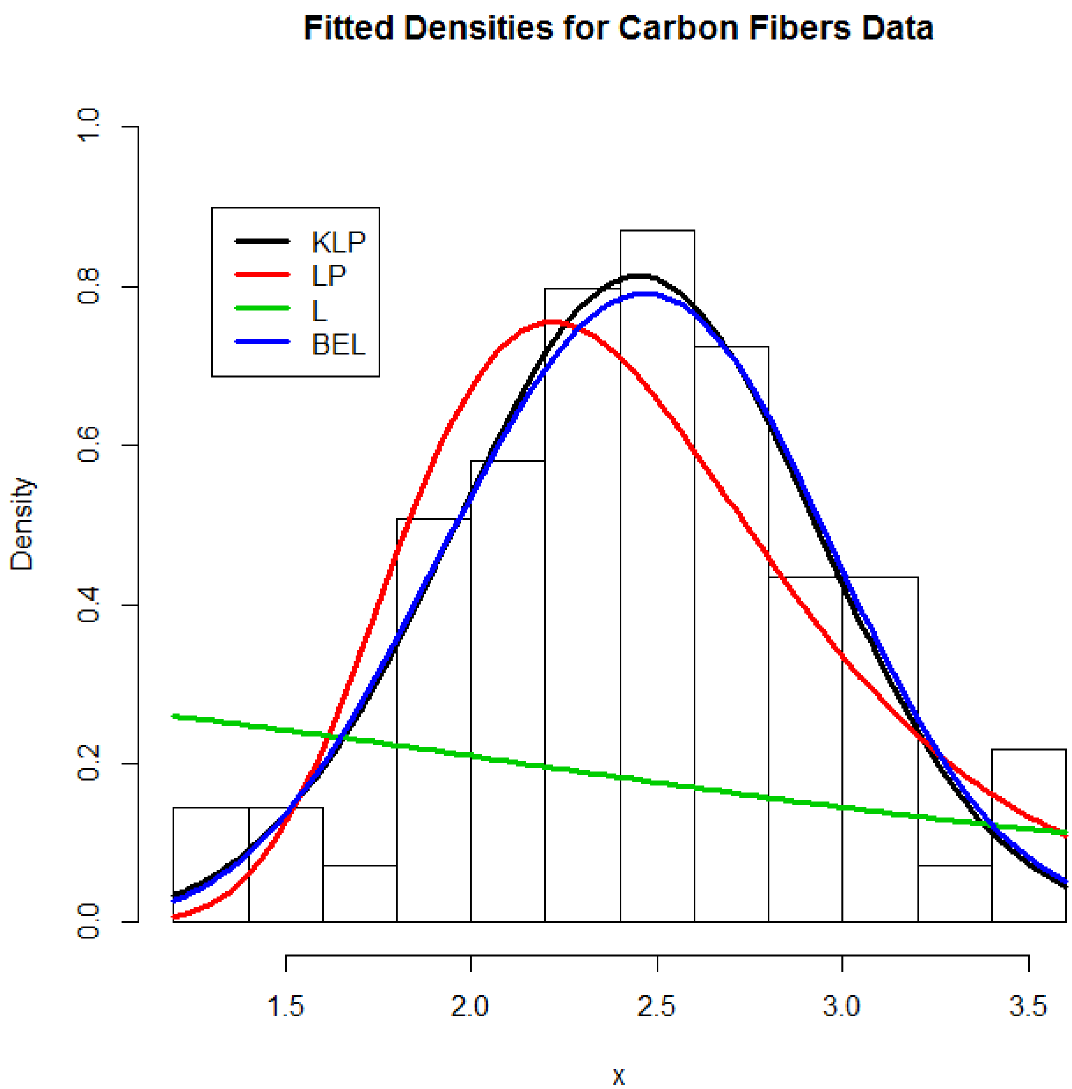

FIGURE 7.5. Histogram and Fitted Density for Carbon Fibers Data

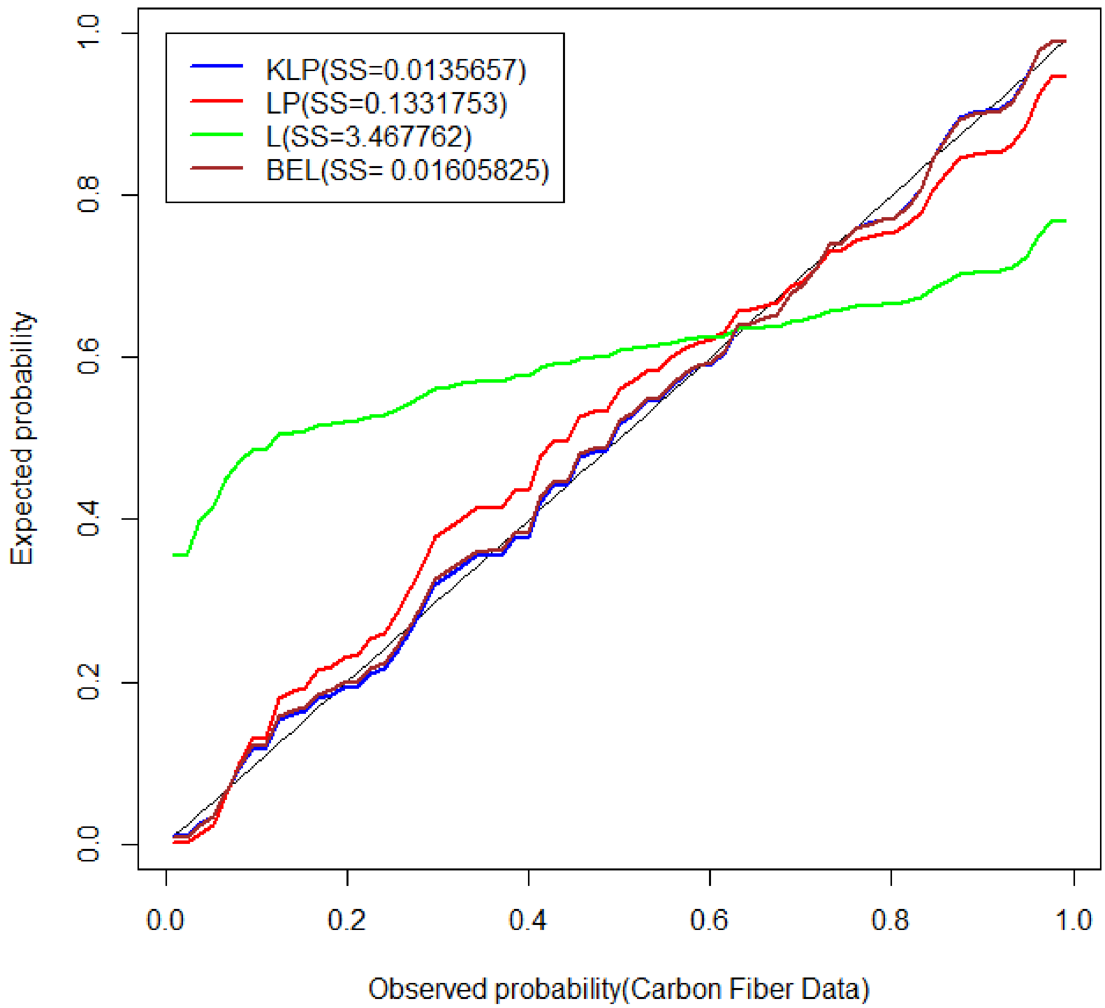

FIGURE 7.6. Probability Plots for Carbon Fibers Data

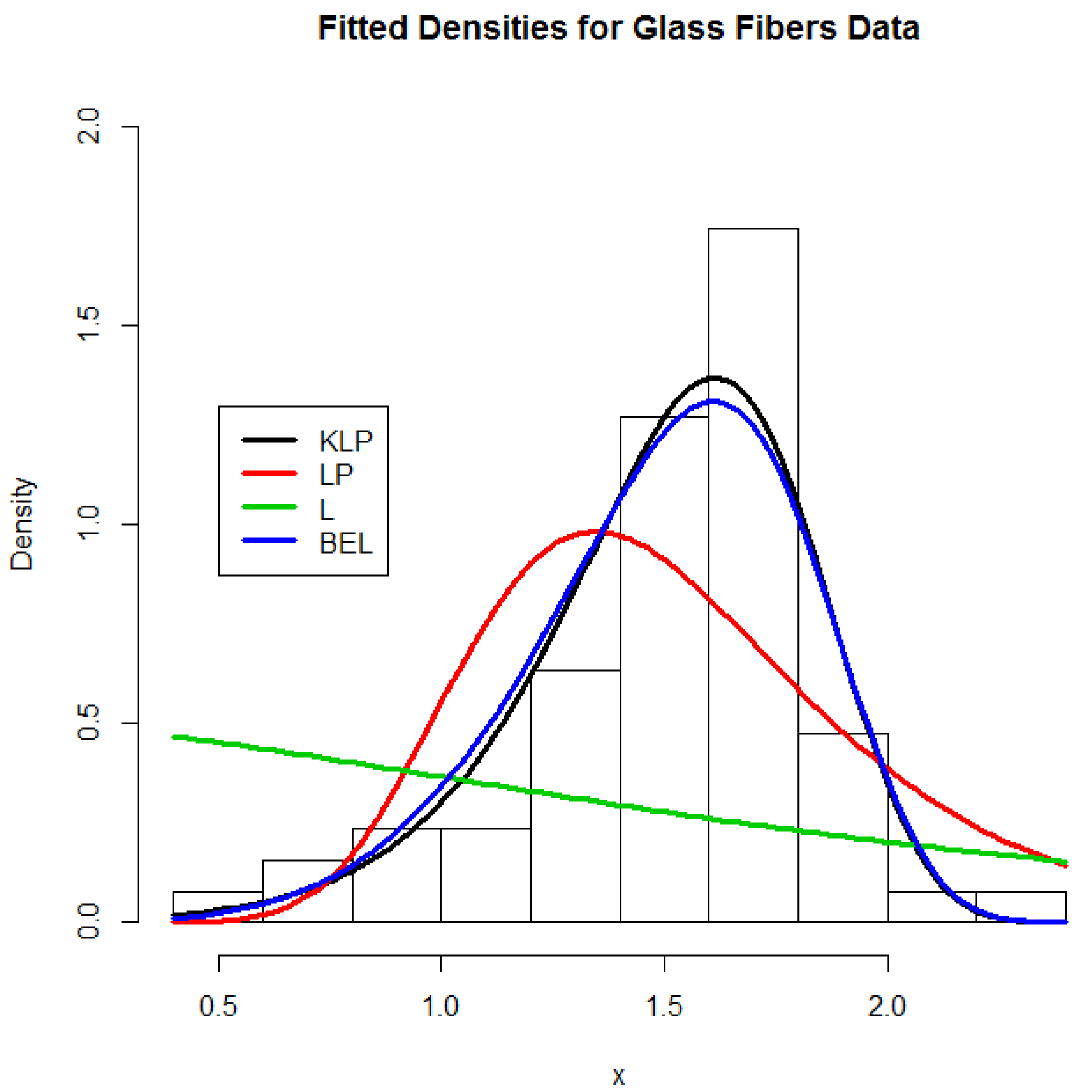

FIGURE 7.7. Histogram and Fitted Density for Glass Fibers Data

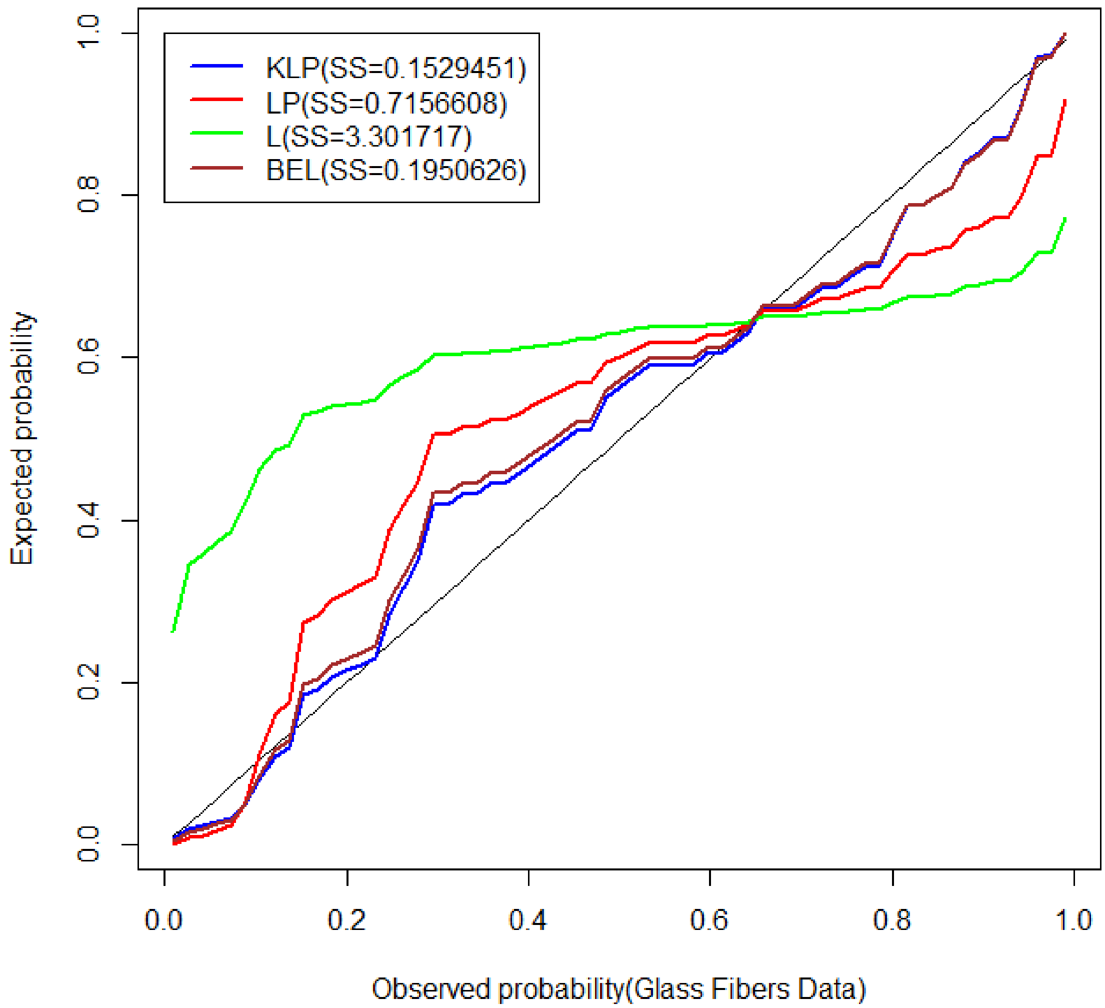

FIGURE 7.8. Probability Plots for Glass Fibers Data

8. Concluding Remarks

We have proposed and presented a new class of lifetime distributions called the Kumaraswamy Lindley-Poisson distribution. This class of distributions has applications in lifetime data analysis. The KLP distribution has the LP, exponentiated Lindley Poisson and Lindley distributions as special cases. The density of the new distribution can be expressed as a linear combination of Lindley Poisson distributions. The KLP distribution has a hazard function that displays flexible behavior. Moments, mean deviations, distribution of order statistics and Rényi entropy were derived. The method of maximum likelihood was used to estimate the model parameters. Finally, KLP distribution is fitted to real data sets in order to illustrate the applicability and usefulness of the distribution.

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