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SOME COMMON FIXED POINT THEOREMS FOR CONTRACTIVE MAPPINGS IN COMPLEX-VALUED B-METRIC SPACES

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Abstract. In this paper, we obtain some common fixed point theorems for two single-valued mappings satisfying rational expressions in complex valued b-metric spaces. Our results improve several well-known conventional results. Also, an example is given to illustrate our obtained result.

1. Introduction

The fixed point theorem, generally known as the Banach contraction mapping principle, appeared in explicit form in Banach's thesis in 1922 [9]. It has applications in different branches of Mathematical analysis and provides the solution of many problems in mathematical analysis. Later, a lot of articles have been dedicated to the improvement and generalization of the Banach's contraction mapping principle in different spaces.

In 1989, Bakhtin [6] introduced the concept of b-metric space as a generalization of metric spaces. Since then many papers have been devoted in this direction [4,7]. A new space called the complex-valued metric space which is more general form than metric spaces has been introduced by Azam et. al [1] and established the existence of fixed point theorems for maps satisfying the contraction condition. In 2012, Rouzkard and Imdad [3] proved some common fixed point theorems satisfying certain rational expressions in complex valued metric spaces which is extended and improved form of the results of Azam et al. [1]. Sintunavarat and Kumam [12] established common fixed point results by replacing constant of contractive condition to control functions.

The concept of complex valued b-metric spaces was introduced in 2013 by Rao et al. [8], which was more general than the well known complex valued metric spaces[1]. In sequel, AA.Mukheimer [2], proved some

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common fixed point theorems of two self mappings satisfying a rational inequality on complex valued b-metric spaces.

In this paper, some common fixed point theorems for a pair of maps under contraction involving rational expressions in the setting of complex valued b-metric spaces are proved. An examples is given to support the usability of our results. The obtained results are generalizations of recent results proved by Azam et al. [1], AA. Mukheimer [2], H.K.Nashine [5], Bhatt et al.[10], Datta et al. [11], Chakkrid Klin-eam and Cholatis Suanoom [13] and Dubey et al.[14].

2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows: $z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2)$.

Consequently, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2), Im(z_1) < Im(z_2)$,:
(ii) $Re(z_1) < Re(z_2), Im(z_1) = Im(z_2)$,:
(iii) $Re(z_1) < Re(z_2), Im(z_1) < Im(z_2)$,:
(iv) $Re(z_1) = Re(z_2), Im(z_1) = Im(z_2)$.:

In particular, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i),(ii) and (iii) is satisfied and we write $z_1 \prec z_2$ if only (iii) is satisfied. Notice that

(a): If $0 \precsim z_1 \precnsim z_2$, then $|z_1| < |z_2|$,
(b): If $z_1 \precsim z_2$ and $z_2 \prec z_3$ then $z_1 \prec z_3$,
(c): If $a, b \in \mathbb{R}$ and $a \leq b$ then $az \precsim bz$ for all $z \in \mathbb{C}$.
The: following definition is recently introduced by Rao et al. [8].

Definition 2.1. Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(i): $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
(ii): $d(x, y) = d(y, x)$;
(iii): $d(x, y) \precsim s[d(x, z) + d(z, y)]$.
The: pair (X,d) is called a complex valued b-metric space.

Example 2.2[8]**.** Let $X = [0, 1]$. Define the mapping $d : X \times X \to \mathbb{C}$ by $d(x, y) = |x - y|^2 + i|x - y|^2$, for all $x, y \in X$.

Then (X, d) is a complex valued b-metric space with $s = 2$.

Definition 2.3[8]**.** Let (X, d) be a complex valued b-metric space.

(i) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A$.

(ii) A point $x \in X$ is called a limit point of a set A whenever for every $0 \prec r \in \mathbb{C}, B(x,r) \cap (A - \{x\}) \neq \phi$.

(iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A.

(iv) A subset $A \subseteq X$ is called closed whenever each element of A belongs to A.

(v) A sub-basis for a Hausdorff topology τ on X is a family $F = \{B(x,r) : x \in X \, and \, 0 \prec r\}$.

Definition 2.4[8]. Let (X,d) be a complex valued b-metric space, $\{x_n\}$be a sequence in X and $x \in X$.

(i) If for every $c \in \mathbb{C}$, with $0 \prec r$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$converges to x and x is the limit point of $\{x_n\}$. We denote this by $lim_{n\to\infty} x_n = x$ or $\{x_n\} \to x \, as \, n \to \infty$.

(ii) If for every $c \in \mathbb{C}$, with $0 \prec r$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$is said to be Cauchy sequence.

(iii) If every Cauchy sequence in X is convergent, then (X,d) is said to be a complete complex valued b-metric space.

Lemma 2.5[8]. Let (X,d) be a complex valued b-metric space and let $\{x_n\}$be a sequence in X. Then $\{x_n\}$converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma 2.6[8]. Let (X,d) be a complex valued b-metric space and let $\{x_n\}$be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.

3. Main Results

In this section, we will prove some common fixed point theorems for the contractive mappings in complex valued b-metric space.

Theorem 3.1. Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S, T : X \to X$ are mappings satisfying:

$$d(Sx,Ty) \precsim Ad(x,y) + \frac{Bd(x,Sx)d(y,Ty)}{1+d(x,y)} + \frac{Cd(y,Sx)d(x,Ty)}{1+d(x,y)}$$
$$+ \frac{Dd(x,Sx)d(x,Ty)}{1+d(x,y)} + \frac{Ed(y,Sx)d(y,Ty)}{1+d(x,y)} - - - (3.1)$$

for all $x, y \in X$, where A, B, C, D and E are nonnegative reals with $A + B + C + 2sD + 2sE < 1$. Then S and T have a unique common fixed in X.

Proof. For any arbitrary point $x_0 \in X$. Define sequence $\{x_n\}$ in X such that

$x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1},\ for\, n = 0, 1, 2, 3, \ldots\ldots - - - (3.2)$

Now, we show that the sequence $\{x_n\}$ is Cauchy. Let $x = x_{2n}$ and $y = x_{2n+1}$ in (3.1), we have $d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1})$

$$\precsim Ad(x_{2n}, x_{2n+1}) + \frac{Bd(x_{2n},Sx_{2n})d(x_{2n+1},Tx_{2n+1})}{1+d(x_{2n},x_{2n+1})}$$
$$+\frac{Cd(x_{2n+1},Sx_{2n})d(x_{2n},Tx_{2n+1})}{1+d(x_{2n},x_{2n+1})}$$
$$+\frac{Dd(x_{2n},Sx_{2n})d(x_{2n},Tx_{2n+1})}{1+d(x_{2n},x_{2n+1})}$$
$$+\frac{Ed(x_{2n+1},Sx_{2n})d(x_{2n+1},Tx_{2n+1})}{1+d(x_{2n},x_{2n+1})}$$
$$\precsim Ad(x_{2n}, x_{2n+1}) + \frac{Bd(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+2})}{1+d(x_{2n},x_{2n+1})}$$
$$+\frac{Dd(x_{2n},x_{2n+1})d(x_{2n},x_{2n+2})}{1+d(x_{2n},x_{2n+1})}$$

which implies that

$$|d(x_{2n+1}, x_{2n+2})| \leq A|d(x_{2n}, x_{2n+1})| + \frac{B|d(x_{2n},x_{2n+1})||d(x_{2n+1},x_{2n+2})|}{|1+d(x_{2n},x_{2n+1})|}$$
$$+\frac{D|d(x_{2n},x_{2n+1})||d(x_{2n},x_{2n+2})|}{|1+d(x_{2n},x_{2n+1})|}.$$

Since $|1 + d(x_{2n}, x_{2n+1})| > |d(x_{2n}, x_{2n+1})|$, so we get

$$|d(x_{2n+1}, x_{2n+2})| \leq A|d(x_{2n}, x_{2n+1})|+B|d(x_{2n+1}, x_{2n+2})|+D|d(x_{2n}, x_{2n+2})|$$

and hence

$$|d(x_{2n+1}, x_{2n+2})| \leq \frac{A+sD}{1-B-sD}|d(x_{2n}, x_{2n+1})|. - - - (3.3)$$

Similarly, we obtain

$$|d(x_{2n+2}, x_{2n+3})| \leq \frac{A+sE}{1-B-sE}|d(x_{2n+1}, x_{2n+2})|. - - - (3.4)$$

Since $A + B + C + 2sD + 2sE < 1\, and\, s \geq 1$ we get $A + B + 2sD < 1\, and\, A + B + 2sE < 1$.

Therefore with $\delta = max\left\{\frac{A+sD}{1-B-sD}, \frac{A+sE}{1-B-sE}\right\} < 1$, and for all $n \geq 0$ and consequently, we have

$$|d(x_{2n+1}, x_{2n+2})| \leq \delta|d(x_{2n}, x_{2n+1})| \leq \delta^2|d(x_{2n-1}, x_{2n})| - - - - - - - - - - - - \leq \delta^{2n+1}|d(x_0, x_1)|.$$

Similarly, we obtain

$$|d(x_{n+1}, x_{n+2})| \leq \delta|d(x_n, x_{n+1})| \leq \delta^2|d(x_{n-1}, x_n)| - - - - - - - - - - - - - - - \leq \delta^{n+1}|d(x_0, x_1)|. - - -(3.5)$$

Thus for any $m > n, m, n \in \mathbb{N}$, we get

$$|d(x_n, x_m)| \leq s|d(x_n, x_{n+1})| + s|d(x_{n+1}, x_m)|$$
$$\leq s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| + s^2|d(x_{n+2}, x_m)|$$
$$\leq s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| + s^3|d(x_{n+2}, x_{n+3})| + s^3|d(x_{n+3}, x_m)|$$

$- -$

$- -$

$$\leq s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| + s^3|d(x_{n+2}, x_{n+3})| + - - - - - - - - - - - - +s^{m-n-2}|d(x_{m-3}, x_{m-2})|+s^{m-n-1}|d(x_{m-2}, x_{m-1})|+s^{m-n}|d(x_{m-1}, x_m)|.$$

By using (3.5), we get

$|d(x_n, x_m)| \leq s\delta^n|d(x_0, x_1)| + s^2\delta^{n+1}|d(x_0, x_1)| + s^3\delta^{n+2}|d(x_0, x_1)|$
$+ - - - - + s^{m-n-2}\delta^{m-3}|d(x_0, x_1)| + s^{m-n-1}\delta^{m-2}|d(x_0, x_1)| + s^{m-n}\delta^{m-1}|d(x_0, x_1)|$
$= \sum_{i=1}^{m-n} s^i\delta^{i+n-1}|d(x_0, x_1)|.$

Therefore,

$$|d(x_n, x_m)| \leq \sum_{i=1}^{m-n} s^{i+n-1}\delta^{i+n-1}|d(x_0, x_1)|$$
$$= \sum_{t=n}^{m-1} s^t\delta^t|d(x_0, x_1)| \leq \sum_{t=n}^{\infty}(s\delta)^t|d(x_0, x_1)|$$
$$= \frac{(s\delta)^n}{1-s\delta}|d(x_0, x_1)|$$

and hence

$|d(x_n, x_m)| \leq \frac{(s\delta)^n}{1-s\delta}|d(x_0, x_1)| \to 0 \, as \, m, n \to \infty.$

Thus $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists some $u \in X$ such that $x_n \to u$ as $n \to \infty$. Assume not, then there exists $z \in X$ such that

$|d(u, Su)| = |z| > 0. - - - (3.6)$

So by using the trianglular inequality and (3.1), we get

$z = d(u, Su) \precsim sd(u, x_{2n+2}) + sd(x_{2n+2}, Su) = sd(u, x_{2n+2}) + sd(Tx_{2n+1}, Su)$
$\precsim sd(u, x_{2n+2}) + sAd(u, x_{2n+1}) + \frac{sBd(u,Su)d(x_{2n+1},Tx_{2n+1})}{1+d(u,x_{2n+1})}$
$+ \frac{sCd(x_{2n+1},Su)d(u,Tx_{2n+1})}{1+d(u,x_{2n+1})} + \frac{sDd(u,Su)d(u,Tx_{2n+1})}{1+d(u,x_{2n+1})} + \frac{sEd(x_{2n+1},Su)d(x_{2n+1},Tx_{2n+1})}{1+d(u,x_{2n+1})}$

which implies that

$|z| = |d(u, Su)| \leq s|d(u, x_{2n+2})| + sA|d(u, x_{2n+1})|$
$+ \frac{sB|d(u,Su)||d(x_{2n+1},x_{2n+2})|}{|1+d(u,x_{2n+1})|}$
$+ \frac{sC|d(x_{2n+1},Su)||d(u,x_{2n+2})|}{|1+d(u,x_{2n+1})|}$
$+ \frac{sD|d(u,Su)||d(u,x_{2n+2})|}{|1+d(u,x_{2n+1})|}$
$+ \frac{sE|d(x_{2n+1},Su)||d(x_{2n+1},x_{2n+2})|}{|1+d(u,x_{2n+1})|}. - - -(3.7)$

Taking the limit of (3.7) as $n \to \infty$, we obtain that $|z| = |d(u, Su)| \leq 0$, a contradiction with (3.6). So $|z| = 0$.

Hence $Su = u$. It follows that similarly $Tu = u$. Therefore, u is common fixed point of S and T.

Finally, to prove the uniqueness of common fixed point, let $u^\star$ is another common fixed point of S and T. Then

$d(u, u^\star) = d(Su, Tu^\star)$

$\precsim Ad(u,u^\star)+\frac{Bd(u,Su)d(u^\star,Tu^\star)}{1+d(u,u^\star)}+\frac{Cd(u^\star,Su)d(u,Tu^\star)}{1+d(u,u^\star)}$
$+\frac{Dd(u,Su)d(u,Tu^\star)}{1+d(u,u^\star)}+\frac{Ed(u^\star,Su)d(u^\star,Tu^\star)}{1+d(u,u^\star)}$.

Taking modulus of the above inequality, we get
$|d(u,u^\star)|\le A|d(u,u^\star)|+\frac{C|d(u^\star,Su)||d(u,Tu^\star)|}{|1+d(u,u^\star)|}$
$=A|d(u,u^\star)|+C|d(u^\star,u)|\frac{|d(u,u^\star)|}{|1+d(u,u^\star)|}$.
Since $|1+d(u,u^\star)|>|d(u,u^\star)|$.
Therefore
$|d(u,u^\star)|<A|d(u,u^\star)|+C|d(u,u^\star)|=(A+C)|d(u,u^\star)|$.
This is contradiction to $A+C<1$. Hence $u=u^\star$which proves the uniqueness of common fixed point in X. This completes the proof.

Corollary 3.2. Let (X,d) be a complete complex valued b-metric space with the coefficient $s\ge 1$ and let $T:X\to X$ be a mapping satisfying:
$d(Tx,Ty)\precsim Ad(x,y)+\frac{Bd(x,Tx)d(y,Ty)}{1+d(x,y)}+\frac{Cd(y,Tx)d(x,Ty)}{1+d(x,y)}$
$+\frac{Dd(x,Tx)d(x,Ty)}{1+d(x,y)}+\frac{Ed(y,Tx)d(y,Ty)}{1+d(x,y)}---(3.8)$
for all $x,y\in X$, where A,B,C,D and E are nonnegative reals with $A+B+C+2sD+2sE<1$. Then T has a unique common fixed in X.

Proof. We can prove this result by applying Theorem 3.1 by setting $S=T$.

Corollary 3.3. Let (X,d) be a complete complex valued b-metric space with the coefficient $s\ge 1$ and let $T:X\to X$ be a mapping satisfying:
$d(T^nx,T^ny)\precsim Ad(x,y)+\frac{Bd(x,T^nx)d(y,T^ny)}{1+d(x,y)}+\frac{Cd(y,T^nx)d(x,T^ny)}{1+d(x,y)}+\frac{Dd(x,T^nx)d(x,T^ny)}{1+d(x,y)}$
$+\frac{Ed(y,T^nx)d(y,T^ny)}{1+d(x,y)}---(3.9)$
for all $x,y\in X$, where A,B,C,D and E are nonnegative reals with $A+B+C+2sD+2sE<1$. Then T has a unique fixed point in X.

Proof. From Corollary 3.2, we obtain $u\in X$ such that
$T^nu=u$.
The uniqueness follows from
$d(Tu,u)=d(TT^nu,T^nu)=d(T^nTu,T^nu)$
$\precsim Ad(Tu,u)+\frac{Bd(Tu,T^nTu)d(u,T^nu)}{1+d(Tu,u)}$
$+\frac{Cd(u,T^nTu)d(Tu,T^nu)}{1+d(Tu,u)}+\frac{Dd(Tu,T^nTu)d(Tu,T^nu)}{1+d(Tu,u)}$
$+\frac{Ed(u,T^nTu)d(u,T^nu)}{1+d(Tu,u)}$
$\precsim Ad(Tu,u)+Cd(u,Tu)\frac{d(Tu,u)}{1+d(Tu,u)}.---(3.10)$

By taking modulus of (3.10), we get

$|d(Tu,u)| \leq A|d(Tu,u)| + C|d(Tu,u)|\frac{|d(Tu,u)|}{|1+d(Tu,u)|}$.

Since $|1 + d(Tu,u)| > |d(Tu,u)|$.

Therefore,

$|d(Tu,u)| < (A + C)|d(Tu,u)|$, a contradiction. So $Tu = u$.

Hence $Tu = T^n u = u$.

Therefore, the fixed point of T is unique. This completes the proof.

Corollary 3.4. Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S,T : X \to X$ are mappings satisfying:

$$d(Sx,Ty) \precsim Ad(x,y) + \frac{Bd(x,Sx)d(y,Ty)}{1+d(x,y)} + \frac{Cd(y,Sx)d(x,Ty)}{1+d(x,y)} + \frac{Dd(x,Sx)d(x,Ty)}{1+d(x,y)} \quad ---(3.11)$$

for all $x, y \in X$, where A, B, C and D are nonnegative reals with $A + B + C + 2sD < 1$. Then S and T have a unique common fixed in X.

Proof. We can prove this result by applying Theorem 3.1 by setting $E = 0$.

Corollary 3.5. Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mapping satisfying:

$$d(Tx,Ty) \precsim Ad(x,y) + \frac{Bd(x,Tx)d(y,Ty)}{1+d(x,y)} + \frac{Cd(y,Tx)d(x,Ty)}{1+d(x,y)} + \frac{Dd(x,Tx)d(x,Ty)}{1+d(x,y)} \quad ---(3.12)$$

for all $x, y \in X$, where A, B, C and D are nonnegative reals with $A + B + C + 2sD < 1$. Then T has a unique fixed point.

Proof. We can prove this result by applying Corollary 3.4 by setting $T = S$ and $E = 0$.

Corollary 3.6. Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S,T : X \to X$ are mappings satisfying:

$$d(Sx,Ty) \precsim Ad(x,y) + \frac{Bd(x,Sx)d(y,Ty)}{1+d(x,y)} + \frac{Cd(y,Sx)d(x,Ty)}{1+d(x,y)} + \frac{Ed(y,Sx)d(y,Ty)}{1+d(x,y)} \quad ---(3.13)$$

for all $x, y \in X$, where A, B, C and E are nonnegative reals with $A + B + C + 2sE < 1$. Then S and T have a unique common fixed in X.

Proof. We can prove this result by applying Theorem 3.1 by setting $D = 0$.

Corollary 3.7. Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mapping satisfying:

$$d(Tx,Ty) \precsim Ad(x,y) + \frac{Bd(x,Tx)d(y,Ty)}{1+d(x,y)} + \frac{Cd(y,Tx)d(x,Ty)}{1+d(x,y)} + \frac{Ed(y,Tx)d(y,Ty)}{1+d(x,y)} - - - (3.14)$$

for all $x, y \in X$, where A, B, C and E are nonnegative reals with $A + B + C + 2sE < 1$. Then T has a unique fixed point.

Proof. We can prove this result by applying Corollary 3.6 by setting $T = S$ and $D = 0$.

Example 3.8. Let $X = \mathbb{C}$. Define a function $d : X \times X \to \mathbb{C}$ such that $d(z_1,z_2) = |x_1 - x_2|^2 + i|y_1 - y_2|^2$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

It is clear that (X,d) is a complex valued b-metric space with $s = 2$.

Now, define two self mappings $S, T : X \to X$ as follows:

$$Tz = T(x+iy) = \begin{cases} 0, \ x,y \in Q \\ 1+i, \ x,y \in Q^c \\ 1, \ x \in Q^c, y \in Q \\ i, \ x \in Q, y \in Q^c \end{cases}$$

such that $S = T$,and $z = x + iy$. Let $x = \frac{1}{\sqrt{2}}$ and $y = 0$: and since $A \in [0,1)$ we have

$$d(Tx,Ty) = d\left(T(\tfrac{1}{\sqrt{2}}),T(0)\right) = d(1,0) = 2 \succ A.\sqrt{2}$$
$$= Ad(\tfrac{1}{\sqrt{2}},0) + \frac{Bd\left(\frac{1}{\sqrt{2}},T(\frac{1}{\sqrt{2}})\right)d(0,T(0))}{1+d(\frac{1}{\sqrt{2}},0)} + \frac{Cd\left(0,T(\frac{1}{\sqrt{2}})\right)d\left(\frac{1}{\sqrt{2}},T(0)\right)}{1+d(\frac{1}{\sqrt{2}},0)} + \frac{Dd\left(\frac{1}{\sqrt{2}},T(\frac{1}{\sqrt{2}})\right),d\left(\frac{1}{\sqrt{2}},T(0)\right)}{1+d(\frac{1}{\sqrt{2}},0)} + \frac{Ed\left(0,T(\frac{1}{\sqrt{2}})\right)d(0,T(0))}{1+d(\frac{1}{\sqrt{2}},0)}.$$

However, notice that $T^n z = 0$ for $n > 1$, so

$$d(T^n x,T^n y) = 0 \precsim Ad(x,y) + \frac{Bd(x,T^n x)d(y,T^n y)}{1+d(x,y)} + \frac{Cd(y,T^n x)d(x,T^n y)}{1+d(x,y)} + \frac{Dd(x,T^n x)d(x,T^n y)}{1+d(x,y)} + \frac{Ed(y,T^n x)d(y,T^n y)}{1+d(x,y)}$$

for all $x, y \in X$, where $A, B, C, D, E \geq 0$ with $A+B+C+2sD+2sE < 1$. So all conditions of Corollary 3.3 are satisfied to get a unique fixed $0\, of\, T$.

Our next theorem is a generalization of Theorem 3.1 of [5] in complex valued b-metric spaces.

Theorem 3.9 Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S,T : X \to X$ are mappings satisfying:

$d(Sx,Ty) \precsim \alpha d(x,y) + \frac{\beta d(y,Ty)d(x,Sx)}{1+d(x,y)} + \gamma[d(x,Sx)+d(y,Ty)]$
$+\delta[d(x,Ty)+d(y,Sx)] ---(3.15)$

for all $x,y \in X$, where $\alpha,\beta,\gamma,\delta$ are nonnegative reals with $\alpha+\beta+2\gamma+2s\delta < 1$. Then S and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X and define sequence $\{x_n\}$ in X such that

$x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1}$ for $n = 0,1,2,.... --- (3.16)$

Now, we show that the sequence $\{x_n\}$ is Cauchy. Let $x = x_{2n}$ and $y = x_{2n+1}$ in (3.15) we have

$d(x_{2n+1},x_{2n+2}) = d(Sx_{2n},Tx_{2n+1})$
$\precsim \alpha d(x_{2n},x_{2n+1}) + \frac{\beta d(x_{2n+1},Tx_{2n+1})d(x_{2n},Sx_{2n})}{1+d(x_{2n},x_{2n+1})}$
$+\gamma[d(x_{2n},Sx_{2n})+d(x_{2n+1},Tx_{2n+1})]+\delta[d(x_{2n},Tx_{2n+1})+d(x_{2n+1},Sx_{2n})]$
$= \alpha d(x_{2n},x_{2n+1}) + \frac{\beta d(x_{2n+1},x_{2n+2})d(x_{2n},x_{2n+1})}{1+d(x_{2n},x_{2n+1})}$
$+\gamma[d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})]$
$+\delta[d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+1})]$
$\vdots$
$\precsim \alpha d(x_{2n},x_{2n+1}) + \frac{\beta d(x_{2n+1},x_{2n+2})d(x_{2n},x_{2n+1})}{1+d(x_{2n},x_{2n+1})}$
$+\gamma[d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})] + s\delta[d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})]$

which implies that

$|d(x_{2n+1},x_{2n+2})| \leq \alpha|d(x_{2n},x_{2n+1})| + \frac{\beta|d(x_{2n+1},x_{2n+2})||d(x_{2n},x_{2n+1})|}{|1+d(x_{2n},x_{2n+1})|}$
$+\gamma[|d(x_{2n},x_{2n+1})|+|d(x_{2n+1},x_{2n+2})|]$
$+s\delta[|d(x_{2n},x_{2n+1})|+|d(x_{2n+1},x_{2n+2})|]$.

Since $|d(x_{2n},x_{2n+1})| \leq |1+d(x_{2n},x_{2n+1})|$,
so we get $|d(x_{2n+1},x_{2n+2})| \leq \alpha|d(x_{2n},x_{2n+1})| + \beta|d(x_{2n+1},x_{2n+2})|$
$+\gamma[|d(x_{2n},x_{2n+1})|+|d(x_{2n+1},x_{2n+2})|]$
$+s\delta[|d(x_{2n},x_{2n+1})|+|d(x_{2n+1},x_{2n+2})|]$

and hence

$|d(x_{2n+1},x_{2n+2})| \leq \left(\frac{\alpha+\gamma+s\delta}{1-\beta-\gamma-s\delta}\right)|d(x_{2n},x_{2n+1})|. --- (3.17)$

Similarly, we obtain

$|d(x_{2n+2},x_{2n+3})| \leq \left(\frac{\alpha+\gamma+s\delta}{1-\beta-\gamma-s\delta}\right)|d(x_{2n+1},x_{2n+2})|. --- (3.18)$

Put $\mu = \frac{\alpha+\gamma+s\delta}{1-\beta-\gamma-s\delta} < 1$, we have

$|d(x_{n+1},x_{n+2})| \leq \mu|d(x_n,x_{n+1})| \leq --- \leq \mu^{n+1}|d(x_0,x_1)|. --- (3.19)$

Thus for any $m > n, m,n \in \mathbb{N}$,

$|d(x_n, x_m)| \leq s|d(x_n, x_{n+1})| + s|d(x_{n+1}, x_m)|$
$\leq s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| + s^2|d(x_{n+2}, x_m)|$
$- -$
$\leq s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| + s^3|d(x_{n+2}, x_{n+3})| + - - -$
$+s^{m-n-1}|d(x_{m-2}, x_{m-1})| + s^{m-n}|d(x_{m-1}, x_m)|.$:

By using (3.19) we get

$|d(x_n, x_m)| \leq s\mu^n|d(x_0, x_1)| + s^2\mu^{n+1}|d(x_0, x_1)| + s^3\mu^{n+2}|d(x_0, x_1)|$
$+ - - - - - - +s^{m-n-1}\mu^{m-2}|d(x_0, x_1)| + s^{m-n}\mu^{m-1}|d(x_0, x_1)|$
$= \sum_{i=1}^{m-n} s^i\mu^{i+n-1}|d(x_0, x_1)|.$

Therefore,

$|d(x_n, x_m)| \leq \sum_{i=1}^{m-n} s^{i+n-1}\mu^{i+n-1}|d(x_0, x_1)|$
$= \sum_{t=n}^{m-1} s^t\mu^t|d(x_0, x_1)| \leq \sum_{t=n}^{\infty}(s\mu)^t|d(x_0, x_1)|$
$= \frac{(s\mu)^n}{1-s\mu}|d(x_0, x_1)|$

and so,

$|d(x_n, x_m)| \leq \frac{(s\mu)^n}{1-s\mu}|d(x_0, x_1)| \to 0$ as $m, n \to \infty$.

This implies that $\{x_n\}$is a Cauchy sequence in X. Since X is complete, there exists some $u \in X$ such that $x_n \to u$ as $n \to \infty$. Let on contrary $u \neq Su$, then there exists $z \in X$ such that

$|d(u, Su)| = |z| > 0. - - - (3.20)$

So by using the triangluar inequality and (3.15), we get

$z = d(u, Su) \precsim sd(u, x_{2n+2}) + sd(x_{2n+2}, Su)$
$= sd(u, x_{2n+2}) + sd(Tx_{2n+1}, Su)$
$\precsim sd(u, x_{2n+2}) + s\alpha d(u, x_{2n+1}) + \frac{s\beta d(x_{2n+1}, Tx_{2n+1})d(u, Su)}{1+d(u, x_{2n+1})}$
$+s\gamma[d(u, Su) + d(x_{2n+1}, Tx_{2n+1})]$
$+s\delta[d(u, Tx_{2n+1}) + d(x_{2n+1}, Su)]$
$= sd(u, x_{2n+2}) + s\alpha d(u, x_{2n+1}) + \frac{s\beta d(x_{2n+1}, x_{2n+2})d(u, Su)}{1+d(u, x_{2n+1})}$
$+s\gamma[d(u, Su) + d(x_{2n+1}, x_{2n+2})] + s\delta[d(u, x_{2n+2}) + d(x_{2n+1}, Su)].$

This implies that

$|z| = |d(u, Su)| \leq s|d(u, x_{2n+2})| + s\alpha|d(x_{2n+1}, u)|$
$+ \frac{s\beta|z||d(x_{2n+1}, x_{2n+2})|}{|1+d(u, x_{2n+1})|}$
$+s\gamma[|z| + |d(x_{2n+1}, x_{2n+2})|]$
$+s\delta[|d(u, x_{2n+2})| + |d(x_{2n+1}, Su)|]. - - - (3.21)$

Taking the limit of (3.21) as $n \to \infty$, we obtain that $|z| = |d(u, Su)| \leq 0$, a contradiction with (3.20). So $|z| = 0 \Rightarrow Su = u$. Similarly, one can show that $u = Tu$.

We now show that $S\, and\, T$ have unique common fixed point. For this, assume that $u^\star$in X is another common fixed point of $S\, and\, T$. Then

$d(u, u^\star) = d(Su, Tu^\star)$

$\precsim \alpha d(u, u^\star) + \frac{\beta d(u,Su)d(u^\star,Tu^\star)}{1+d(u,u^\star)} + \gamma[d(u, Su) + d(u^\star, Tu^\star)]$

$+\delta[d(u, Tu^\star) + d(u^\star Su)]$.

So that $|d(u, u^\star)| \leq \alpha|d(u, u^\star)| + \frac{\beta|d(u,Su)||d(u^\star,Tu^\star)|}{|1+d(u,u^\star)|}$

$+\gamma[|d(u, Su)| + |d(u^\star, Tu^\star)|]+\delta[|d(u, Tu^\star)| + |d(u^\star, Su)|]$

$= (\alpha + 2\delta)|d(u, u^\star)|$.

This implies that $u^\star = u$, which proves the uniqueness of common fixed point in X. This completes the proof.

Corollary 3.10. Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mapping satisfying:

$d(Tx, Ty) \precsim \alpha d(x, y) + \frac{\beta d(y,Ty)d(x,Tx)}{1+d(x,y)}$

$+\gamma[d(x, Tx) + d(y, Ty)]$

$+\delta[d(x, Ty) + d(y, Tx)] - - - (3.22)$

for all $x, y \in X$, where $\alpha, \beta, \gamma, \delta$ are nonnegative reals with $\alpha+\beta+2\gamma+2s\delta < 1$. Then T has a unique common fixed point in X.

Proof. We can prove this result by applying Theorem 3.9 with $S = T$.

Corollary 3.11 Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mapping satisfying (for some fixed n):

$d(T^n x, T^n y) \precsim \alpha d(x, y) + \frac{\beta d(y,T^n y)d(x,T^n x)}{1+d(x,y)}$

$+\gamma[d(x, T^n x) + d(y, T^n y)]$

$+\delta[d(x, T^n y) + d(y, T^n x)] - --(3.23)$

for all $x, y \in X$, where $\alpha, \beta, \gamma, \delta$ are nonnegative reals with $\alpha+\beta+2\gamma+2s\delta < 1$. Then T has a unique fixed point in X.

Proof. From Corollary 3.10, we obtain $u \in X$ such that

$T^n u = u$.

The uniqueness follows from

$d(Tu, u) = d(TT^n u, T^n u) = d(T^n Tu, T^n u)$

$\precsim \alpha d(Tu, u) + \frac{\beta d(u,T^n u)d(Tu,T^n Tu)}{1+d(Tu,u)}$

$+\gamma[d(Tu, T^n Tu) + d(u, T^n u)]$

$+\delta[d(Tu, T^n u) + d(u, T^n Tu)]$

$\precsim \alpha d(Tu,u) + \frac{\beta d(u,u) d(Tu,TT^n u)}{1+d(Tu,u)}$
$+\gamma[d(Tu,TT^n u) + d(u,u)]$
$+\delta[d(Tu,u) + d(u,TT^n u)]$
$= (\alpha + 2\delta) d(Tu,u). - - - (3.24)$

By taking modulus of (3.24) and since $\alpha+2\delta < 1$, we obtain $|d(Tu,u)| \leq (\alpha + 2\delta)|d(Tu,u)| < |d(Tu,u)|$,a contradiction.

So $Tu = u$. Hence

$Tu = T^n u = u.$

Therefore, the fixed point of T is unique. This completes the proof.

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