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# ON THE OSCILLATION OF THIRD ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The aim of this paper is to study asymptotic properties of class of the third-order neutral delay differential equation

$$\left(r_2(t)\left[\left(r_1(t)\left[z'(t)\right]^{\alpha_1}\right)'\right]^{\alpha_2}\right)' + q(t) f(x(g(t))) = 0, t \geq t_0,$$

where $z(t) = x(t) + p(t) x(\tau(t))$. By different methods, new sufficient conditions which insure that the solution is oscillatory or converges to zero are established. Some examples are considered to illustrate the main results.

## 1. Introduction

In this paper, we are concerned with the oscillation and the asymptotic behavior of solutions of the third-order nonlinear neutral differential equations with delayed argument

$$\left(r_2(t)\left[\left(r_1(t)\left[z'(t)\right]^{\alpha_1}\right)'\right]^{\alpha_2}\right)' + q(t) f(x(g(t))) = 0, \quad t \geq t_0, \tag{1.1}$$

where $z(t) = x(t) + p(t) x(\tau(t))$. In the sequel we will assume that the following conditions are satisfied:

- $(C_1)$ $p, q, \tau, g \in C([t_0, \infty), \mathbb{R})$, $\alpha_1$ and $\alpha_2$ are a quotient of odd positive integers, $q$ is positive, $\beta = \alpha_1\alpha_1$, $0 \leq p(t) \leq p < 1$, $\tau(t) \leq t$, $g(t) \leq t$, $g'(t) > 0$, $\lim_{t\to\infty} \tau(t) = \infty$ and $\lim_{t\to\infty} g(t) = \infty$;
- $(C_2)$ $r_i \in C([t_0, \infty), (0, \infty))$, $\int_{t_0}^{\infty} r_i^{-1/\alpha_i}(t)\, dt = \infty$, $i = 1, 2$;
- $(C_3)$ $f \in C(\mathbb{R}, \mathbb{R})$, $\frac{f(x)}{x^\beta} \geq k > 0$ for $x \neq 0$.

By a solution of Eq. (1.1), we mean a non-trivial real function $x(t) \in C([t_x, \infty))$, $t_x \geq t_0$, which has the properties $z(t)$, $r_1(t)(z'(t))^{\alpha_1}$ and $r_2(t)((r_1(t)(z'(t))^{\alpha_1})')^{\alpha_2}$ are continuously differentiable for all $t \in [t_x, \infty)$ and satisfies (1.1) on $[t_x, \infty)$. We consider only those solutions $x(t)$ of (1.1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for any $T \geq t_x$. A solution of Eq. (1.1) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory.

In the last few years, there has been increasing interest in obtaining sufficient conditions for the oscillation and nonoscillation of solutions of second/third-order neutral delay differential

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equations, see for example [1] - [12] and the references quoted therein. Special cases of equation (1.1) include the delay equation

$$\left(r_2(t)\left((x(t)+p(t)\,x(\tau(t)))''\right)^{\alpha_2}\right)'+q(t)\,f(x(g(t)))=0. \tag{1.2}$$

The oscillatory behavior of solutions of (1.2) have been discussed in a number of studies and we refer the reader, for example, to the monographs by Baculikova [3], Dzurina [7], Su [16] and Thandapani [17]. Recently, by comparison with some first differential equations whose oscillatory characters are known, Elabbasy [8] established some general oscillation criteria for third order neutral delay differential equation

$$\left(r_2(t)\left[\left(r_1(t)\,[x'(t)]^{\alpha_1}\right)'\right]^{\alpha_2}\right)'+q(t)\,f(x(g(t)))=0. \tag{1.3}$$

In this paper, our aim is devoted to third-order neutral delay equations. We have greatly less results for third order equation than for the first or second order equations. We generalize techniques in [3], [17] to obtain oscillation criteria for new equation that extend and generalize earlier ones. As well as, we used the Riccati-technique on equation (1.1) and examples are considered to illustrate the main results.

## 2. Main Results

In this section, we will establish some new oscillation criteria for solutions of the Eq. (1.1). For the sake of convenience, we introduce the following notation:

$$E_0(t)=z(t),\ E_i(t)=r_i(t)\left(\frac{d}{dt}E_{i-1}(t)\right)^{\alpha_i},\ i=1,2,$$

$$R_u(t)=\frac{1}{r_1^{1/\alpha_1}(t)}\left(\int_u^t\frac{ds}{r_2^{1/\alpha_2}(s)}\right)^{1/\alpha_1}\text{ and }\overline{R}_u(t)=\int_u^t R_u(s)ds.$$

First, we state and prove some useful lemmas.

**Lemma 2.1.** *Let $x(t)$ be a positive solution of Eq. (1.1). Then $z(t)$ has only one of the following two properties eventually:*

(i) $z(t)>0, z'(t)>0$ and $\frac{d}{dt}E_1(t)>0$,
(ii) $z(t)>0, z'(t)<0$ and $\frac{d}{dt}E_1(t)>0$.

*Proof.* Let $x(t)$ be a positive solution of Eq. (1.1). From $(C_1)$, there exists a $t_1\geq t_0$ such that $x(t)>0$, $x(\tau(t))>0$ and $x(g(t))>0$ for $t\geq t_1$. Then $z(t)>0$ and Eq. (1.1) implies that

$$\frac{d}{dt}E_2(t)=-q(t)\,f(x(g(t)))\leq 0.$$

Hence, $E_2(t)$ is a non-increasing function and of one sign. We claim that $E_2(t) > 0$ for $t \geq t_1$. Suppose that $E_2(t) < 0$ for $t \geq t_2 \geq t_1$, then there exists a $t_3 \geq t_2$ and constant $K_1 > 0$ such that

$$\frac{d}{dt}E_1(t) < -K_1(r_2(t))^{-1/\alpha_2},$$

for $t \geq t_3$. By integrating the last inequality from $t_3$ to $t$, we get

$$E_1(t) < E_1(t_3) - K_1 \int_{t_3}^{t} (r_2(s))^{-1/\alpha_2} ds.$$

Letting $t \to \infty$, from $(C_2)$, we have $\lim\limits_{t\to\infty} E_1(t) = -\infty$. Then there exists a $t_4 \geq t_3$ and constant $K_2 > 0$ such that

$$z'(t) < -K_2(r_1(t))^{-1/\alpha_1},$$

for $t \geq t_4$. By integrating this inequality from $t_4$ to $\infty$ and using $(C_2)$, we get $\lim\limits_{t\to\infty} z(t) = -\infty$, which contradicts $z(t) > 0$. Now we have $E_2(t) > 0$ for $t \geq t_1$. Therefore, $E_1(t)$ is increasing function. Thus $(i)$ or $(ii)$ holds for $z(t)$ eventually. □

**Lemma 2.2.** *Let $x(t)$ be a positive solution of Eq. (1.1), and $z(t)$ has the property $(ii)$. Assume that*

$$\int_{t_0}^{\infty} \frac{1}{r_1^{1/\alpha_1}(v)} \left( \int_{v}^{\infty} \frac{1}{r_2^{1/\alpha_2}(u)} \left( \int_{u}^{\infty} q(s)ds \right)^{1/\alpha_2} du \right)^{1/\alpha_1} dv = \infty. \tag{2.1}$$

*Then the solution $x(t)$ of Eq. (1.1) is converges to zero as $t \to \infty$.*

*Proof.* Let $x(t)$ be a positive solution of Eq. (1.1). Since $z(t)$ satisfies the property $(ii)$, we get

$$\lim_{t\to\infty} z(t) = \gamma \geq 0.$$

Now, we shall prove that $\gamma = 0$. Let $\gamma > 0$, then we have $\gamma < z(t) < \gamma + \varepsilon$ for all $\varepsilon > 0$ and $t$ enough large. Choosing $\varepsilon < \frac{1-p}{p}\gamma$, we obtain

$$\begin{aligned} x(t) &= z(t) - p(t)x(\tau(t)) \\ &> \gamma - pz(\tau(t)) \\ &> L(\gamma + \varepsilon) > Lz(t), \end{aligned}$$

where $L = \frac{\gamma - p(\gamma+\varepsilon)}{\gamma+\varepsilon} > 0$. Hence, from Eq. (1.1) and $(C_3)$, we have

$$\begin{aligned} \frac{d}{dt}E_2(t) &\leq -kq(t)\,x^{\beta}(g(t)) \\ &< -kL^{\beta}q(t)\,z^{\beta}(g(t)) \\ &< -kL^{\beta}\gamma^{\beta}q(t). \end{aligned}$$

Integrating this inequality from $t$ to $\infty$, we get

$$\frac{d}{dt}E_1(t) > -k^{1/\alpha_2}L^{\alpha_1}\gamma^{\alpha_1}\frac{1}{r_2^{1/\alpha_2}(t)}\left(\int_t^{\infty} q(s)ds\right)^{1/\alpha_2}.$$

Integrating again from $t$ to $\infty$, we obtain

$$z'(t) < -C\frac{1}{r_1^{1/\alpha_1}(t)}\left(\int_t^\infty \frac{1}{r_2^{1/\alpha_2}(u)}\left(\int_u^\infty q(s)ds\right)^{1/\alpha_2} du\right)^{1/\alpha_1},$$

where $C = k^{1/\beta}L\gamma > 0$. Integrating the last inequality from $t_1$ to $\infty$, we have

$$z(t_1) > C\int_{t_1}^\infty \frac{1}{r_1^{1/\alpha_1}(v)}\left(\int_v^\infty \frac{1}{r_2^{1/\alpha_2}(u)}\left(\int_u^\infty q(s)ds\right)^{1/\alpha_2} du\right)^{1/\alpha_1} dv.$$

This contradicts to the condition (2.1), then $\lim_{t\to\infty} z(t) = 0$, which implies that $\lim_{t\to\infty} x(t) = 0$. □

**Lemma 2.3.** *Let $x(t)$ be a positive solution of Eq. (1.1), and $z(t)$ has the property $(i)$. Then, we have*

$$\frac{d}{dt}E_2(t) \leq -kq(t)(1-p(g(t)))^\beta z^\beta(g(t)), \tag{2.2}$$

$$z'(g(t)) \geq E_2^{1/\beta}(t)R_{t_0}(g(t)) \tag{2.3}$$

*and*

$$\overline{R}_{t_0}^\beta(g(t))\frac{E_2(t)}{z^\beta(g(t))} \leq 1. \tag{2.4}$$

*Proof.* Let $x(t)$ be a positive solution of Eq. (1.1). From $(C_1)$, there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(g(t)) > 0$ for $t \geq t_1$. Since $z(t)$ satisfies the property $(i)$, then we get

$$\begin{aligned} x(t) &= z(t) - p(t)x(\tau(t)) \\ &\geq (1-p(t))z(t). \end{aligned}$$

Thus, by Eq. (1.1) and $(C_3)$, we have

$$\begin{aligned} \frac{d}{dt}E_2(t) &\leq -kq(t)x^\beta(g(t)) \\ &\leq -kq(t)(1-p(g(t)))^\beta z^\beta(g(t)) < 0. \end{aligned}$$

Now, from property $(i)$, there exists a $T \geq t_0$ such that

$$E_1(t) = E_1(T) + \int_T^t \frac{E_2^{1/\alpha_2}(s)}{r_2^{1/\alpha_2}(s)}ds.$$

Since $\frac{d}{dt}E_2(t) < 0$, we obtain

$$E_1(t) \geq E_2^{1/\alpha_2}(t)\int_T^t \frac{1}{r_2^{1/\alpha_2}(s)}ds.$$

This implies that

$$z'(t) \geq E_2^{1/\beta}(t) R_T(t). \tag{2.5}$$

Since $g\left(t\right) \leq t$, we have

$$z'(g(t)) \geq E_2^{1/\beta}(t) R_T\left(g(t)\right).$$

By integrating the inequality (2.5) from $T$ to $t$ and using $\frac{d}{dt}E_2(t) < 0$, we get

$$\begin{aligned} z\left(t\right) &\geq z\left(T\right) + E_2^{1/\beta}(t) \int_T^t R_T(s) ds \\ &\geq E_2^{1/\beta}(t) \overline{R}_T(t). \end{aligned}$$

Thus, we get

$$z\left(g\left(t\right)\right) \geq E_2^{1/\beta}(t) \overline{R}_T(g\left(t\right)),$$

and so

$$\overline{R}_T^{\beta}(g\left(t\right)) \frac{E_2\left(t\right)}{z^{\beta}\left(g\left(t\right)\right)} \leq 1.$$

This completes the proof. $\square$

Now, for simplicity, we introduce the following notation:

$$P = \liminf_{t\to\infty} \overline{R}_{t_0}^{\beta}\left(g\left(t\right)\right) \int_t^{\infty} \theta\left(s\right) ds$$

and

$$Q = \limsup_{t\to\infty} \frac{1}{\overline{R}_{t_0}\left(g\left(t\right)\right)} \int_{t_0}^t \overline{R}_{t_0}^{\beta+1}\left(g\left(s\right)\right) \theta\left(s\right) ds,$$

where $\theta\left(t\right) = kq\left(t\right)\left(1 - p\left(g\left(t\right)\right)\right)^{\beta}$. Moreover for $z\left(t\right)$ satisfying property (i), we define

$$\omega\left(t\right) = \frac{E_2\left(t\right)}{z^{\beta}\left(g\left(t\right)\right)} \tag{2.6}$$

and

$$l = \liminf_{t\to\infty} \overline{R}_{t_0}^{\beta}\left(g\left(t\right)\right) \omega\left(t\right), \quad U = \limsup_{t\to\infty} \overline{R}_{t_0}^{\beta}\left(g\left(t\right)\right) \omega\left(t\right). \tag{2.7}$$

**Lemma 2.4.** *Let $x(t)$ be a positive solution of Eq. (1.1).*

(1) *Let $P < \infty, Q < \infty$ and $z\left(t\right)$ satisfies property (i). If*

$$\lim_{t\to\infty} \overline{R}_{t_0}(t) = \infty, \tag{2.8}$$

*then*

$$P \leq l - l^{\frac{1+\beta}{\beta}} \text{ and } P + Q \leq 1 \tag{2.9}$$

(2) *If $P = \infty$ or $Q = \infty$, then $z\left(t\right)$ does not have property (i).*

*Proof.* **Part (1)**. Let $x(t)$ be a positive solution of Eq. (1.1) and $z\left(t\right)$ satisfies property (i). By Lemma 2.3, we have that (2.2), (2.3) and (2.4) hold. From the definition of $\omega\left(t\right)$ , we see that $\omega\left(t\right)$ is positive and satisfies

$$\omega'\left(t\right)=\frac{\frac{d}{dt}E_{2}\left(t\right)}{z^{\beta}\left(g\left(t\right)\right)}-\beta\frac{E_{2}\left(t\right)}{z^{\beta+1}\left(g\left(t\right)\right)}z'\left(g\left(t\right)\right)g'\left(t\right).$$

Thus, from (2.2) and (2.3), there exists a $T\geq t_{0}$ such that

$$\omega'\left(t\right)\leq-kq\left(t\right)\left(1-p\left(g\left(t\right)\right)^{\beta}-\beta\frac{E_{2}^{\frac{1+\beta}{\beta}}\left(t\right)}{z^{\beta+1}\left(g\left(t\right)\right)}R_{T}(g\left(t\right))g'\left(t\right),$$

for $t\geq T$. This implies that

$$\omega'\left(t\right)\leq-\theta\left(t\right)-\beta R_{T}(g\left(t\right))g'\left(t\right)\omega^{\frac{1+\beta}{\beta}}\left(t\right). \tag{2.10}$$

From (2.4), we get

$$\overline{R}_{T}(g\left(t\right))\omega\left(t\right)\leq1,$$

which with (2.8) gives

$$\lim_{t\rightarrow\infty}\omega\left(t\right)=0. \tag{2.11}$$

On the other hand, from the definition of $\omega\left(t\right),l$ and $U$, we see that

$$0\leq l\leq U\leq1. \tag{2.12}$$

Now, we prove that the first inequality in (2.9) holds. Let $\varepsilon>0$, then from the definitions of $P$ and $l$,we can choose $t_{2}\geq T$ sufficiently large that

$$\overline{R}_{T}^{\beta}\left(g\left(t\right)\right)\int_{t}^{\infty}\theta\left(s\right)ds\geq P-\varepsilon\ \text{ and }\overline{R}_{T}^{\beta}\left(g\left(t\right)\right)\omega\left(t\right)\geq l-\varepsilon\quad\text{for }t\geq t_{2}.$$

By integrating (2.10) from $t$ to $\infty$ and using (2.11), we have

$$\omega\left(t\right)\geq\int_{t}^{\infty}\theta\left(s\right)ds+\beta\int_{t}^{\infty}R_{T}(g\left(s\right))g'\left(s\right)\omega^{\frac{1+\beta}{\beta}}\left(s\right)ds. \tag{2.13}$$

Multiplying the above inequality by $\overline{R}_{T}^{\beta}\left(g\left(t\right)\right)$, we obtain

$$\begin{aligned}\overline{R}_{T}^{\beta}\left(g\left(t\right)\right)\omega\left(t\right) &\geq \overline{R}_{T}^{\beta}\left(g\left(t\right)\right)\int_{t}^{\infty}\theta\left(s\right)ds\\ &\quad+\beta\overline{R}_{T}^{\beta}\left(g\left(t\right)\right)\int_{t}^{\infty}\frac{R_{T}(g\left(s\right))g'\left(s\right)}{\overline{R}_{T}^{\beta+1}\left(g\left(s\right)\right)}\left(\overline{R}_{T}^{\beta}\left(g\left(s\right)\right)\omega\left(s\right)\right)^{\frac{1+\beta}{\beta}}ds\\ &\geq \left(P-\varepsilon\right)+\left(l-\varepsilon\right)^{\frac{1+\beta}{\beta}}\overline{R}_{T}^{\beta}\left(g\left(t\right)\right)\int_{t}^{\infty}\frac{\beta R_{T}(g\left(s\right))g'\left(s\right)}{\overline{R}_{T}^{\beta+1}\left(g\left(s\right)\right)}ds\\ &\geq \left(P-\varepsilon\right)+\left(l-\varepsilon\right)^{\frac{1+\beta}{\beta}}.\end{aligned}$$

Taking the limit inferior on both sides as $t\rightarrow\infty$, we get

$$l\geq\left(P-\varepsilon\right)+\left(l-\varepsilon\right)^{\frac{1+\beta}{\beta}}.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the desired result

$$P \leq l - l^{\frac{1+\beta}{\beta}}.$$

Next, we prove the second inequality in part (1). Multiplying (2.10) by $\overline{R}_T^{\beta+1}(g(t))$ and integrating it from $t_2$ to $t$, we obtain

$$\begin{aligned}\int_{t_2}^{t} \overline{R}_T^{\beta+1}(g(s))\,\omega'(s)\,ds &\leq -\int_{t_2}^{t} \overline{R}_T^{\beta+1}(g(s))\,\theta(s)\,ds \\ &\quad -\beta\int_{t_2}^{t} R_T(g(s))g'(s)\left(\overline{R}_T^{\beta}(g(s))\,\omega(s)\right)^{\frac{1+\beta}{\beta}} ds.\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}\overline{R}_T^{\beta+1}(g(t))\,\omega(t) &\leq \overline{R}_T^{\beta+1}(g(t_2))\,\omega(t_2) - \int_{t_2}^{t} \overline{R}_T^{\beta+1}(g(s))\,\theta(s)\,ds \\ &\quad + \int_{t_2}^{t} R_T(g(s))g'(s)\left((\beta+1)V - \beta V^{\frac{1+\beta}{\beta}}\right) ds,\end{aligned}$$

where $V = \overline{R}_T^{\beta}(g(s))\,\omega(s)$. Using the inequality

$$a\phi - b\phi^{\frac{1+\beta}{\beta}} \leq \frac{\beta^{\beta}}{(\beta+1)^{\beta+1}} a^{\beta+1} b^{-\beta} \quad \text{for } a \geq 0, b > 0 \text{ and } \phi \geq 0, \tag{2.14}$$

with $\phi = V, a = (\beta+1)$ and $b = \beta$. Thus, we get

$$\begin{aligned}\overline{R}_T^{\beta+1}(g(t))\,\omega(t) &\leq \overline{R}_T^{\beta+1}(g(t_2))\,\omega(t_2) - \int_{t_2}^{t} \overline{R}_T^{\beta+1}(g(s))\,\theta(s)\,ds \\ &\quad + \overline{R}_T(g(t)) - \overline{R}_T(g(t_2)).\end{aligned}$$

It follows that

$$\begin{aligned}\overline{R}_T^{\beta}(g(t))\,\omega(t) &\leq \frac{\overline{R}_T^{\beta+1}(g(t_2))\,\omega(t_2)}{\overline{R}_T(g(t))} - \frac{1}{\overline{R}_T(g(t))}\int_{t_2}^{t} \overline{R}_T^{\beta+1}(g(s))\,\theta(s)\,ds \\ &\quad + 1 - \frac{\overline{R}_T(g(t_2))}{\overline{R}_T(g(t))}.\end{aligned}$$

Taking the limit superior on both sides as $t \to \infty$ and using (2.8) we get

$$U \leq 1 - Q.$$

Thus, from (2.12), we have

$$P \leq l - l^{\frac{1+\beta}{\beta}} \leq l \leq U \leq 1 - Q, \tag{2.15}$$

which completes the proof of Part (1).

In **Part (2)**, Let $x(t)$ is a positive solution of Eq. (1.1). We shall proof that $z(t)$ does not have property (i). On the contrary, we assume that $P = \infty$. Then, from (2.13), we get

$$\overline{R}_T^{\beta}(g(t))\,\omega(t) \geq \overline{R}_T^{\beta}(g(t)) \int_{t}^{\infty} \theta(s)\,ds.$$

Taking the lim inf of both sides as $t \to \infty$, we get in view of (2.12) that

$$1 \geq l \geq P = \infty.$$

This is a contradiction. Now we admit that $Q = \infty$. Then by (2.15), $U = -\infty$, which contradicts (2.12). The proof is complete. □

**Theorem 2.1.** *Assume that (2.1) and (2.8) hold. If*

$$P = \liminf_{t \to \infty} \overline{R}_{t_0}^{\beta}\left(g\left(t\right)\right) \int_t^{\infty} \theta\left(s\right) ds > \frac{\beta^{\beta}}{\left(\beta+1\right)^{\beta+1}}. \tag{2.16}$$

*Then every solution of Eq. (1.1) is either oscillatory or tends to zero as $t \to \infty$.*

*Proof.* Let $x$ be a non-oscillatory solution of Eq. (1.1). Without loss of generality we may assume that $x(t) > 0$. If $P = \infty$, then by Lemma 2.4 $z\left(t\right)$ does not have property (i). That is, $z\left(t\right)$ satisfies property (ii). Therefore, from Lemma 2.2, we have $\lim_{t \to \infty} x(t) = 0$.

Now, Let $P < \infty$. By Lemma 2.1, we have that $z(t)$ has the property (i) or the property (ii). If $z(t)$ has the property (ii), from Lemma 2.2, we obtain $\lim_{t \to \infty} x(t) = 0$. Next, we assume that for $z\left(t\right)$ property (i) holds. Let $\omega$ and $l$ be defined by (2.6) and (2.7), respectively. Then from Lemma 2.4, we have $P \leq l - l^{\frac{1+\beta}{\beta}}$. Using inequality (2.14) with $\phi = l$ and $a = b = 1$, we get that

$$P \leq \frac{\beta^{\beta}}{\left(\beta+1\right)^{\beta+1}},$$

which contradicts (2.16). The proof is complete. □

**Theorem 2.2.** *Assume that (2.1) and (2.8) hold. If*

$$P + Q > 1, \tag{2.17}$$

*then every solution of Eq. (1.1) is either oscillatory or tends to zero as $t \to \infty$.*

*Proof.* Let $x$ be a non-oscillatory solution of Eq. (1.1). Without loss of generality we may assume that $x(t) > 0$. If $P = \infty$ or $Q = \infty$, then by Lemma 2.4 $z\left(t\right)$ does not have property (i). That is, $z\left(t\right)$ must satisfy property (ii). Then from Lemma 2.2, we get that $\lim_{t \to \infty} x(t) = 0$.

Next, Let $P < \infty$ and $Q < \infty$. By Lemma 2.1, we have that $z(t)$ has the property (i) or (ii). If for $z(t)$ property (ii) holds, then exactly as above we are led by Lemma 2.2 to $\lim_{t \to \infty} x(t) = 0$. Now, we assume that $z\left(t\right)$ has the property (i). Then from Lemma 2.4, we have $P + Q \leq 1$ which contradicts (2.16). The proof is complete. □

As a consequence of Theorem 2.2, we have the following results.

**Corollary 2.1.** *Assume that (2.1) and (2.8) hold. If*

$$\limsup_{t\to\infty} \frac{1}{\overline{R}_{t_0}(g(t))} \int_{t_0}^{t} \overline{R}_{t_0}^{\beta+1}(g(s))\,\theta(s)\,ds > 1,$$

*then every solution of Eq. (1.1) is either oscillatory or tends to zero as* $t \to \infty$.

**Example 2.1.** *Consider the third-order neutral delay differential equation*

$$\left(t\left(\left(\frac{1}{t}\left(z'(t)\right)^{1/3}\right)'\right)^{3}\right)' + \frac{\lambda}{t^{6}} x\left(\frac{t}{2}\right) = 0, \quad \lambda > 0, \tag{2.18}$$

*where* $z(t) = x(t) + \frac{1}{2}x\left(\frac{t}{3}\right)$ *and* $t \geq 1$. *We note that* $\beta = 1$ *and* $f(x) = x$. *If we choose* $k = 1$. *Then, we obtain*

$$\overline{R}_{t_0}(t) = \frac{3}{2}\left(t^{5} - 3t^{13/3} + 3t^{11/3} - t^{3}\right) \quad \text{and} \quad \theta(t) = \frac{\lambda}{2}\frac{1}{t^{6}}.$$

*Hence, It easy to see that (2.1) and (2.8) hold and*

$$\liminf_{t\to\infty} \overline{R}_{t_0}^{\beta}(g(t)) \int_{t}^{\infty} \theta(s)\,ds = \frac{27\lambda}{2560}.$$

*Thus, by Theorem 2.1, if* $\lambda > \frac{640}{27}$, *we have that every solution of Eq. (2.18) is either oscillatory or tends to zero.*

**Example 2.2.** *Consider the third order delay differential equation*

$$\left(t\left(\left(x(t) + \frac{1}{3}x\left(\frac{t}{2}\right)\right)''\right)^{3}\right)' + \frac{a}{t^{6}} x^{3}\left(\frac{t}{2}\right) = 0, \quad a > 0, \tag{2.19}$$

*According to Corollary 1 in* [3], *every nonoscillatory solution of Eq. (2.19) converges to zero provided that*

$$a > \frac{9^{3}}{2} = 364.5$$

*If we choose* $k = 1$, *then we conclude that (2.1) and (2.8) are satisfied and (2.16) hold for*

$$a > \frac{625}{8} = 78.125$$

*Hence, by Theorem 2.1, every solution of equation (2.19) is either oscillatory or tends to zero if* $a > 78.125$. *Then, our results supplement and improve the results obtained in* [3].

In the following Theorem, we are concerned with the oscillation of solutions of Eq. (1.1) by using a Riccati transformation technique.

**Theorem 2.3.** *Let (2.1) holds. Assume that there exists a positive function* $\rho(t)$*such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\rho(s)\theta(s) - \frac{\beta^{\beta}}{(\beta+1)^{\beta+1}} \left(\frac{\rho'(s)}{\rho(s)}\right)^{\beta+1} \eta(s)\right) ds = \infty, \tag{2.20}$$

*where* $\eta(t) = \rho(t)\left(\beta R_{t_0}(g(t))\, g'(t)\right)^{-\beta}$ *. Then every solution of Eq. (1.1) is either oscillatory or tends to zero as* $t \to \infty$*.*

*Proof.* Let $x$ be a non-oscillatory solution of Eq. (1.1). Without loss of generality we may assume that $x(t) > 0$. By Lemma 2.1, we have that $z(t)$ has the property (i) or the property (ii). If $z(t)$ has the property (ii), from Lemma 2.2, we obtain $\lim_{t\to\infty} x(t) = 0$. Next, let $z(t)$ satisfies the property (i). By Lemma 2.3, we have that (2.2) and (2.3) hold. Now, We define

$$\widetilde{\omega}(t) = \rho(t)\frac{E_2(t)}{z^{\beta}(g(t))}.$$

By differentiating and using (2.2) and (2.3) we get

$$\widetilde{\omega}'(t) \le -\rho(t)\theta(t) + \frac{\rho'(t)}{\rho(t)}\widetilde{\omega}(t) - \eta^{-1/\beta}(t)\widetilde{\omega}^{\frac{\beta+1}{\beta}}(t).$$

Using inequality (2.14) with $\phi = \widetilde{\omega}$, $a = \frac{\rho'}{\rho}$ and $b = \eta^{-1/\beta}$, we obtain

$$\frac{\rho'}{\rho}\widetilde{\omega} - \eta^{-1/\beta}\widetilde{\omega}^{\frac{\beta+1}{\beta}} \le \frac{\beta^{\beta}}{(\beta+1)^{\beta+1}}\left(\frac{\rho'}{\rho}\right)^{\beta+1}\eta.$$

Therefore, we get

$$\widetilde{\omega}'(t) \le -\rho(t)\theta(t) + \frac{\beta^{\beta}}{(\beta+1)^{\beta+1}}\left(\frac{\rho'(t)}{\rho(t)}\right)^{\beta+1}\eta(t).$$

By integrating the above inequality from $t_2$ to $t$ we have

$$\widetilde{\omega}(t) \le \widetilde{\omega}(t_2) - \int_{t_2}^{t}\left(\rho(s)\theta(s) - \frac{\beta^{\beta}}{(\beta+1)^{\beta+1}}\left(\frac{\rho'(s)}{\rho(s)}\right)^{\beta+1}\eta(s)\right)ds.$$

Taking the superior limit as $t \to \infty$ and using (2.20), we get $\widetilde{\omega}(t) \to -\infty$, which contradicts that $\widetilde{\omega}(t) > 0$. This completes the proof of Theorem 2.3. □

**Example 2.3.** *Consider the third order neutral delay differential equation*

$$\left(t\left(\left(\frac{1}{t}\left(z'(t)\right)^{1/3}\right)'\right)^{5}\right)' + \frac{1}{t}x^{5/3}(t-1)\left(x^{2}(t-1)+2\right) = 0, \tag{2.21}$$

*where* $z(t) = x(t) + \frac{1}{2}x(t-1)$ *and* $t > 1$*. Choose* $\rho(t) = 1$ *and* $k = 2$*. It is easy to see that the conditions (2.1) and (2.20) are hold. Then, from Theorem 2.3, every nonoscillatory solution of Eq. (2.21) tends to zero as* $t \to \infty$*.*

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