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SOME FIXED POINT THEOREMS ON GENERALIZED METRIC SPACES

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ABSTRACT. In this paper, we establish fixed point theorems for mappings satisfying a modified $\gamma-\psi-$contractive mappings in generalized metric spaces. Moreover, the effectiveness of our work is validated with the help of a suitable example.

1. Introduction and preliminaries

In this section, we give some useful definitions and lemmas that will be needed in the sequel.

Definition 1. [2] *Let X be a nonempty set and let $\rho : X \times X \to [0, \infty)$ satisfy the following conditions for all x, $y \in X$ and all distinct u, $v \in X$ each of them different from x and y.*

(1) $\rho(x, y) = 0$ *if and only if* $x = y$,
(2) $\rho(x, y) = \rho(y, x)$,
(3) $\rho(x, y) \leq \rho(x, u) + \rho(u, v) + \rho(v, y)$.

Then the map ρ is called generalized metric and abbreviated as GM. Here, the pair (X, ρ) is called generalized metric space and abbreviated as GMS.

Definition 2. [2] *Let (X, ρ) a g.m.s and $\{x_n\}$ be sequence in X.*

(1) *$\{x_n\}$ is called g.m.s convergent to a limit x if and only if $\rho(x_n, x) \to 0$ as $n \to \infty$,*
(2) *$\{x_n\}$ is called g.m.s Cauchy sequence if and only if for every $\varepsilon > 0$ there exists positive integer $N(\varepsilon)$ such that $\rho(x_n, x_m) < \varepsilon$ for all $n > m > N(\varepsilon)$.*
(3) *A rectangular metric spaces (X, ρ) is called complete if every g.m.s Cauchy sequence is g.m.s convergent.*
(4) *A mapping $T : (X, \rho) \to (X, \rho)$ is continuous if for any sequence $\{x_n\}$ in X such that $\rho(x_n, x) \to 0$ as $n \to \infty$, we have $\rho(Tx_n, Tx) \to 0$ as $n \to \infty$.*

Lemma 1. [6] *Let (X, ρ) be a g.m.s., and let $\{x_n\}$ be a Cauchy sequence in X such that $x_n \neq x_m$ whenever $n \neq m$. Then $\{x_n\}$ can converge to at most one point.*

Lemma 2. [6] *Let (X, ρ) be a g.m.s, and let $\{x_n\}$ be a sequence in X with distinct elements ($x_n \neq x_m$ for $n \neq m$). Suppose that $\rho(x_n, x_{n+1})$ and $\rho(x_n, x_{n+2})$ tend to 0 as $n \to \infty$ and*

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that $\{x_n\}$ *is not a Cauchy sequence. Then there exists* $\varepsilon > 0$ *and two sequences* $\{m_k\}$ *and* $\{n_k\}$ *of positive integers such that* $n_k > m_k > k$ *and the following sequences*

$$\rho(x_{m_k}, x_{n_k}), \; \rho(x_{m_k}, x_{n_{k+1}}), \; \rho(x_{m_{k-1}}, x_{n_k}), \; \rho(x_{m_{k-1}}, x_{n_{k+1}}) \tag{1.1}$$

tend to ε *as* $k \to \infty$.

In 2012, Samet et al. introduced the notion of $\alpha-$*admissible* mappings as follows.

Definition 3. [1] *Let* $T : X \to X$ *and* $\alpha : X \times X \to [0, \infty)$ *be given mappings. We say that* T *is* $\alpha-$*admissible if for all* x, $y \in X$, *we have*

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

In 2014, Popescu [3] investigated the notion of *triangular* $\alpha-$*orbital admissible* as follows.

Definition 4. [3] *Let* $T : X \to X$ *be a mapping and* $\alpha : X \times X \to [0, \infty)$ *be a function. We say that* T *is* $\alpha-$*orbital admissible if*

$$\alpha(x, Tx) \geq 1 \Rightarrow \alpha\left(Tx, T^2x\right) \geq 1.$$

Definition 5. [3] *An* $\alpha-$*orbital admissible map* T *is said to be triangular* $\alpha-$*orbital admissible if*

$$\alpha(x, y) \geq 1 \text{ and } \alpha(y, Ty) \geq 1 \text{ imply } \alpha(x, Ty) \geq 1.$$

Lemma 3. [3] *Let* $T : X \to X$ *be a triangular* $\alpha-$*orbital admissible mapping. Assume that there exists* $x_1 \in X$ *such that* $\alpha(x_1, Tx_1) \geq 1$. *Define a sequence* $\{x_n\}$ *by* $x_{n+1} = Tx_n$. *Then we* $\alpha(x_n, x_m) \geq 1$ *for all* m, $n \in \mathbb{N}$ *with* $n < m$.

Definition 6. [5] *Let* T *be a self-mapping on a metric space* (X, ρ) *and let* α, $\eta : X \times X \to [0, \infty)$ *be two functions. We say that* T *is an* $\alpha-$*admissible with respect to* η *mapping if*

$$x, y \in X, \; \alpha(x, y) \geq \eta(x, y) \Rightarrow \alpha(Tx, Ty) \geq \eta(Tx, Ty).$$

Remark 1. [4] *Define the mapping* $\gamma : X \times X \to [0, \infty)$ *by*

$$\gamma(x, y) = \begin{cases} 1 & \text{if } \alpha(x, y) \geq \eta(x, y), \\ 0 & \text{otherwise} \end{cases}.$$

Also, Bergiz and Karapinar [4] *showed that*

$$\gamma(x, Tx)\gamma(y, Ty) = \begin{cases} 1 & \text{if } \alpha(x, Tx)\alpha(y, Ty) \geq \eta(x, Tx)\eta(y, Ty), \\ 0 & \text{otherwise} \end{cases}.$$

Clearly, if T *is an* $\alpha-$*admissible with respect to* η, *then* T *is* $\gamma-$*admissible.*

In this paper, we establish fixed point theorems for mappings satisfying a modified $\gamma-\psi-$contractive mappings in generalized metric spaces. Moreover, the effectiveness of our work is validated with the help of a suitable example.

2. Main Results

We denote by Ψ the set of functions $\psi : [0,+\infty) \to [0,+\infty)$ satisfying the following hypotheses:

(1) ψ is continuous and nondecreasing,
(2) $\psi(t) = 0$ if and only if $t = 0$.

We denote by Φ^* the set of functions $\varphi : [0,+\infty) \to [0,+\infty)$ satisfying the following hypotheses:

(1) $\liminf_{t\to r^+} \varphi(t) > 0$ for all $r > 0$,
(2) $\varphi(t) = 0$ if and only if $t = 0$.

Theorem 1. *Let (X,ρ) be a g.m.s, and let T be a mapping. Assume that for $\psi \in \Psi$ and $\varphi \in \Phi^*$,*

$$\begin{aligned} x,\ y &\in X,\ \gamma(x,Tx)\gamma(y,Ty) \geq 1 \\ &\Rightarrow \psi(\rho(Tx,Ty)) \leq \psi(\rho(x,y)) - \varphi(\rho(x,y)). \end{aligned} \tag{2.1}$$

Also suppose that the following assertions hold:
(i) T is triangular $\gamma-$orbital admissible;
(ii) there exists $x_0 \in X$ such that $\gamma(x_0,Tx_0) \geq 1$;
(iii) T is continuous or for any sequence $\{x_n\}$ in X with $\gamma(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = x$, we have $\gamma(x,Tx) \geq 1$ for all $n \in \mathbb{N}$.
Then T has a fixed point $u \in X$ and $\rho(u,u) = 0$.

Proof. From (ii) let $x_0 \in X$, construct the sequence $\{x_n\}$ as $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x = x_n$ is a fixed point for T. Assume furher that $x_{n+1} \neq x_n$ for each $n \in \mathbb{N}$. By (ii) if there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq \eta(x_0,Tx_0)$, then this x_0 satisfies also $\gamma(x_0,Tx_0) \geq 1$. By Lemma 3, we have $\gamma(x_n,x_{n+1}) \geq 1$. Clearly,

$$\alpha(x_n,Tx_n)\alpha(x_{n+1},Tx_{n+1}) \geq \eta(x_n,Tx_n)\eta(x_{n+1},Tx_{n+1}).$$

Consequently, we have

$$\gamma(x_n,Tx_n)\gamma(x_{n+1},Tx_{n+1}) \geq 1. \tag{2.2}$$

Since T is *triangular $\gamma-$orbital admissible*, by Lemma 3, we have that

$$\gamma(x_n,x_m) \geq 1, \quad \forall\ m < n \in \mathbb{N}. \tag{2.3}$$

Then,

$$\alpha(x_n,Tx_n)\alpha(x_m,Tx_m) \geq \eta(x_n,Tx_n)\eta(x_m,Tx_m).$$

Thus, we have

$$\gamma(x_n,Tx_n)\gamma(x_m,Tx_m) \geq 1. \tag{2.4}$$

By (2.1), we have

$$\psi\left(\rho\left(x_n,x_{n+1}\right)\right) = \psi\left(\rho\left(Tx_{n-1},Tx_n\right)\right) \leq \psi\left(\rho\left(x_{n-1},x_n\right)\right) - \varphi\left(\rho\left(x_{n-1},x_n\right)\right). \tag{2.5}$$

From (2.5), using the monotone property of function ψ, and $\varphi\left(\rho\left(x_{n-1},x_n\right)\right) > 0$, we get

$$\rho\left(x_n,x_{n+1}\right) < \rho\left(x_{n-1},x_n\right) \text{ for all } n \in \mathbb{N}. \tag{2.6}$$

By (2.6), it follows that the sequence of positive reals $\{\rho\left(x_n,x_{n+1}\right)\}$ is nonincreasing and eventually, there exists $a \geq 0$ such that $\lim_{n\to\infty}\rho\left(x_n,x_{n+1}\right) = a$. We claim that $\lim_{n\to\infty}\rho\left(x_n,x_{n+1}\right) = a = 0$.

Suppose, to the contrary, that $a > 0$. Letting $n \to \infty$ in (2.5) and by the continuity of ψ and the property (1) of function $\varphi \in \Phi^*$, we have

$$\psi\left(a\right) \leq \psi\left(a\right) - \liminf_{\rho\left(x_n,x_{n+1}\right)\to a^+}\varphi\left(a\right) < \psi\left(a\right),$$

a contradiction. Then

$$\lim_{n\to\infty}\rho\left(x_n,x_{n+1}\right) = 0. \tag{2.7}$$

Analogously, we shall prove that $\lim_{n\to\infty}\rho\left(x_n,x_{n+2}\right) = 0$. By (2.1), we have

$$\psi\left(\rho\left(x_n,x_{n+2}\right)\right) = \psi\left(\rho\left(Tx_{n-1},Tx_{n+1}\right)\right) \leq \psi\left(\rho\left(x_{n-1},x_{n+1}\right)\right) - \varphi\left(\rho\left(x_{n-1},x_{n+1}\right)\right). \tag{2.8}$$

From (2.8), using the monotone property of function ψ, and $\varphi\left(\rho\left(x_{n-1},x_{n+1}\right)\right) > 0$, we get

$$\rho\left(x_n,x_{n+2}\right) < \rho\left(x_{n-1},x_{n+1}\right) \text{ for all } n \in \mathbb{N}. \tag{2.9}$$

By (2.9), it follows that the sequence of positive reals $\{\rho\left(x_{n-1},x_{n+1}\right)\}$ is monotone decreasing and eventually, there exists $b \geq 0$ such that $\lim_{n\to\infty}\rho\left(x_{n-1},x_{n+1}\right) = b$. We claim that $\lim_{n\to\infty}\rho\left(x_{n-1},x_{n+1}\right) = b = 0$.

Suppose, to the contrary, that $b > 0$. Letting $n \to \infty$ in (2.8) and by the continuity of ψ and the property (1) of function $\varphi \in \Phi^*$, we have

$$\psi\left(b\right) \leq \psi\left(b\right) - \liminf_{\rho\left(x_{n-1},x_{n+1}\right)\to b^+}\varphi\left(b\right) < \psi\left(b\right),$$

a contradiction. On the other hand, the continuity of ψ yields that

$$b = \lim_{n\to\infty}\rho\left(x_{n-1},x_{n+1}\right) = 0 = \lim_{n\to\infty}\rho\left(x_n,x_{n+2}\right). \tag{2.10}$$

Assume that $x_n = x_m$ for some $m, n \in \mathbb{N}$, $m < n$

$$\begin{aligned} \psi(\rho(x_m, x_{m+1})) &= \psi(\rho(x_n, x_{n+1})) \\ &\leq \psi(\rho(x_{n-1}, x_n)) - \varphi(\rho(x_{n-1}, x_n)) \\ &< \psi(\rho(x_{n-1}, x_n)) \\ &\leq \psi^{n-m}(\rho(x_m, x_{m+1})) \\ &< \psi(\rho(x_m, x_{m+1})), \end{aligned}$$

a contradiction. Therefore, all elements of the sequence $\{x_n\}$ are distinct.

In order to prove that $\{x_n\}$ is a Cauchy sequence in (X, ρ), suppose that it is not. Then by Lemma 2, from (2.7) and (2.10), we claim that there exists $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and (1.1) tends to ε as $k \to \infty$. Regarding (2.1) and (2.4), we find that

$$\begin{aligned} \psi\left(\rho\left(x_{m_k}, x_{n_{k+1}}\right)\right) &= \psi\left(\rho\left(Tx_{m_{k-1}}, Tx_{n_k}\right)\right) \\ &\leq \psi\left(\rho\left(x_{m_{k-1}}, x_{n_k}\right)\right) - \varphi\left(\rho\left(x_{m_{k-1}}, x_{n_k}\right)\right). \end{aligned} \tag{2.11}$$

Letting $n \to \infty$ in (2.11) and by the continuity of ψ and the property (1) of function $\varphi \in \Phi^*$, we have

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \liminf_{\rho\left(x_{m_{k-1}}, x_{n_k}\right) \to \varepsilon^+} \varphi(\varepsilon) < \psi(\varepsilon),$$

which is a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence. Since (X, ρ) is a complete g.m.s., there exists $u \in X$ such that $\lim_{n\to\infty} \rho(x_n, u) = 0$. We suppose that T is continuous, then we have

$$\lim_{n\to\infty} \rho(Tx_n, u) = \lim_{n\to\infty} \rho(x_{n+1}, Tu) = 0.$$

From Lemma 1, we obtain that $Tu = u$. On the other hand, in view of (2.2) and $x_n \to u$ as $n \to \infty$, so

$$\alpha(u, Tu) \geq \eta(u, Tu),$$

which implies

$$\alpha(x_n, x_{n+1})\, \alpha(u, Tu) \geq \eta(x_n, x_{n+1})\, \eta(u, Tu).$$

This implies also

$$\gamma(x_n, Tx_n)\, \gamma(u, Tu) \geq 1.$$

By (2.1) we have

$$\begin{aligned} \psi(\rho(x_{n+1}, Tu)) &= \psi(\rho(Tx_n, Tu)) \\ &\leq \psi(\rho(x_n, u)) - \varphi(\rho(x_n, u)). \end{aligned}$$

Now, by using the properties of ψ and φ and taking $\liminf$ as $n \to \infty$, we obtain

$$\psi(\rho(u, Tu)) = \lim_{n\to\infty} \psi(\rho(x_{n+1}, Tu)) = 0.$$

Then $\rho(u,Tu)=0$. In this way $u=Tu$. Therefore, we obtain that T has a fixed point $u\in X$ and $\rho(u,u)=0$. □

In consequence of Theorem 1, we may indicate the following corollary. Taking $\psi(t)=t$ and $\varphi(t)=(1-k)t$ in Theorem 1, we have following.

Corollary 1. *Let (X,ρ) be a g.m.s, and let T be a mapping. Assume that there exists $k\in[0,1)$ and such that for all $x,\ y\in X$*

$$\begin{aligned} x,\ y &\in X,\ \gamma(x,Tx)\gamma(y,Ty)\geq 1 \\ &\Rightarrow \rho(Tx,Ty)\leq k\rho(x,y). \end{aligned} \tag{2.12}$$

Also suppose that the following assertions hold:

(i) T is triangular $\gamma-$orbital admissible;

(ii) there exists $x_0\in X$ such that $\gamma(x_0,Tx_0)\geq 1$;

(iii) T is continuous or for any sequence $\{x_n\}$ in X with $\gamma(x_n,x_{n+1})\geq 1$ for all $n\in\mathbb{N}$, and $\lim_{n\to\infty}x_n=x$, we have $\gamma(x,Tx)\geq 1$ for all $n\in\mathbb{N}$.

Then T has a fixed point $u\in X$ and $\rho(u,u)=0$.

We give an example which is inspired by Example 39 of [7].

Example 1. *Let $X=\{0,1,2,3\}$. Define $\rho: X\times X\to[0,\infty)$ as follows:*

$\rho(x,y)=\rho(y,x)$ for all $x,\ y\in X$ and $\rho(x,y)=0$ if and only if $y=x$. Further, let

$$\begin{aligned} \rho(0,2) &= \rho(0,3)=\rho(2,3)=2; \\ \rho(0,1) &= \rho(1,2)=4; \\ \rho(1,3) &= 1. \end{aligned}$$

Then it easy to show that (X,ρ) is a complete g.m.s., but (X,ρ) is not a metric space since the triangle inequality does not hold for all $x,\ y,\ z\in X$:

$$4=\rho(1,2)>\rho(1,3)+\rho(3,2)=1+2=3.$$

Define $T:X\to X,\ T(x)=\begin{cases} 1 & \text{if } x\neq 2 \\ 3 & \text{if } x=2 \end{cases}$ *and* $\gamma(x,y)=\begin{cases} 1 & \text{if } x,\ y\in X-\{2\} \\ \frac{5}{6} & \text{otherwise} \end{cases}$.

Firstly, we will prove that

(1) *T is triangular $\gamma-$orbital admissible;*

(2) *there exists $x_0\in X$ such that $\gamma(x_0,Tx_0)\geq 1$;*

(3) *for any sequence $\{x_n\}$ in X with $\gamma(x_n,x_{n+1})\geq 1$ for all $n\in\mathbb{N}$, and $\lim_{n\to\infty}x_n=x$, i have $\gamma(x,Tx)\geq 1$ for all $n\in\mathbb{N}$;*

(4) *T has a fixed point $u\in X$ and $\rho(u,u)=0$.*

Proof. 1. (a) Let $x \in X$ such that $\gamma(x, Tx) \geq 1$ implies $\gamma(Tx, T^2x) \geq 1$. Then, by the definiton of γ, we have $x \in X - \{2\}$, therefore, we obtain

$$\begin{aligned} \gamma(0, T0) &= \gamma(0,1) \geq 1 \text{ and } \gamma\left(T0, T^2 0\right) = \gamma(1,1) \geq 1; \\ \gamma(1, T1) &= \gamma(1,1) \geq 1 \text{ and } \gamma\left(T1, T^2 1\right) = \gamma(1,1) \geq 1; \\ \gamma(3, T3) &= \gamma(3,1) \geq 1 \text{ and } \gamma\left(T3, T^2 3\right) = \gamma(1,1) \geq 1. \end{aligned}$$

We have also shown that T is $\gamma - orbital\ admissible$.

(b) Let x, $y \in X$ such that $\gamma(x, y) \geq 1$ and $\gamma(y, Ty) \geq 1$ imply $\gamma(x, Ty) \geq 1$. Again the definition of γ gives x, $y \in X - \{2\}$, hence we obtain

$$\begin{aligned} \gamma(0,1) &\geq 1 \text{ and } \gamma(1, T1) \geq 1 \text{ imply } \gamma(0, T1) \geq 1; \\ \gamma(0,3) &\geq 1 \text{ and } \gamma(3, T3) \geq 1 \text{ imply } \gamma(0, T3) \geq 1; \\ \gamma(1,3) &\geq 1 \text{ and } \gamma(3, T3) \geq 1 \text{ imply } \gamma(1, T3) \geq 1. \end{aligned}$$

Thereby, (a) and (b) imply that T is *triangular* $\gamma - orbital\ admissible$.

2. Taking $x_0 = 1$, we have $\gamma(1, T1) = \gamma(1,1) \geq 1$.

3. Let $\{x_n\}$ be sequence in X such that $\gamma(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = x$. By the definiton of γ, for each n, $x_n \in X - \{2\} = \{0,1,3\}$. Since $\{0,1,3\}$ is closed, we get that $x \in \{0,1,3\}$. Thus, we have $\gamma(x_n, x) = 1$ for each n.

4. Clearly, T has a fixed point $1 \in X$. Thus, by the definiton of γ, we obtain $\gamma(1,1) = 1$.

Now, we claim that there exists $k \in [0,1)$ and such that for all x, $y \in X$

$$\begin{aligned} x,\ y &\in X,\ \gamma(x, Tx)\,\gamma(y, Ty) \geq 1 \\ &\Rightarrow \rho(Tx, Ty) \leq k\rho(x, y). \end{aligned}$$

Firstly, $\gamma(x, Tx)\,\gamma(y, Ty) \geq 1$ implies $x, y \in X - \{2\}$.

Also, let x, $y \in X$ with $x \neq y$ and consider the following possible cases.

Case 1. If $x, y \in \{0,1,3\}$, then $\rho(Tx, Ty) = \rho(1,1) = 0$ and thus (2.12) trivially holds.

Case 2. If $x = 2$, $y \in \{0,1,3\}$, then $\rho(Tx, Ty) = \rho(3,1) = 1$. Let $k = 0.6$.

If $y = 0$, then

$$1 = \rho(T2, T0) \leq k\rho(2,0) = 0.6.2 = 1.2.$$

If $y = 1$, then

$$1 = \rho(T2, T1) \leq k\rho(2,1) = 0.6.4 = 2.4.$$

If $y = 3$, then

$$1 = \rho(T2, T3) \leq k\rho(2,3) = 0.6.2 = 1.2.$$

Case 3. Let $x \in \{0,1,3\}$, $y = 2$. Since ρ is symmetric, thus (2.12) holds trivially by Case 2.

In this way, inequality (2.12) is satisfied. Hence all the conditions of Corollary 1 are satisfied. $\square$

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