

ON BOUNDARY VALUE PROBLEMS OF HIGHER ORDER FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES

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ABSTRACT. This paper is devoted to study the existence and uniqueness of solutions of fractional integro-differential equations involving Caputo derivative with boundary value conditions of higher order in Banach spaces. A new generalized singular type Gronwall's inequality is given by us to obtain priori bounds. New sufficient conditions for the existence and uniqueness of solution are established by means of fractional calculus, fixed point theorems and Hölder inequality. Examples are provided to illustrate the main results.

1. INTRODUCTION

Fractional differential equations is a generalization of ordinary differential equations and integration to arbitrary non-integer orders. In the last few decades, fractional order models are found to be more adequate than integer order models for some real world problems. There has been a significant progress in the investigation of fractional differential and partial differential equations in the recent years; see the monographs of Kilbas et al. [10], Miller and Ross [15], Podlubny [16], Samko et al. [17] and the references given therein. Recently, some basic theory for the initial value problems of fractional differential equations involving a Riemann-Liouville differential operator of order $\alpha \in (0, 1)$ has been developed by Lakshmikantham and Vatsala in [11, 12, 13, 14]. Fractional differential equations have been proved to be valuable tools in the modelling of many natural phenomena in various fields of engineering, physics, chemistry, aerodynamics, electrodynamics of complex medium, see for example [8, 10]. In [23], authors studied a hyperplastic and fractional derivative viscoelastic model to describe infant brain tissue under conditions consistent with the development of hydrocephalus. Agarwal et al. [2] established sufficient conditions for the existence and uniqueness of solutions for various classes of initial and boundary value problem for fractional differential equations and inclusions involving the Caputo fractional derivative in finite dimensional spaces. Recently, some fractional differential equations and optimal controls in Banach spaces were studied by Balachandran and Park [3], El-Borai [5], Henderson and Ouahab [6], Hernandez et al. [7], Wang et al. [20] and Wang et al. [21, 22].

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Karthikeyan and Trujillo [9], Wang et al. [19] and Yang et al. [24] have extended the work in [1] from real line \mathbb{R} to the abstract Banach space X by using more general assumptions on the nonlinear function f . Very recently, authors [4] have studied boundary value problems for fractional integro-differential equations

$$\begin{cases} {}^c D^\alpha y(t) = f(t, y(t), (Sy)(t)), t \in J = [0, T], \alpha \in (n-1, n), \\ y(0) = y_0, y'(0) = y_0^1, y''(0) = y_0^2, \dots, y^{(n-2)}(0) = y_0^{n-2}, \\ y^{(n-1)}(T) = y_T, \end{cases}$$

where ${}^c D^\alpha$ is the Caputo fractional derivative of order α , $f : J \times X \times X \rightarrow X$, and S is a linear integral operator given by $(Sy)(t) = \int_0^t k(t, s)y(s)ds$, where $k \in C(J \times J, \mathbb{R}^+)$.

We are motivated by the works in [9, 19, 24] and influenced by Chalishajar and Karthikeyan [4]. Our aim in this paper is to consider the following more general boundary value problem for fractional integro-differential equation

$$(1) \quad \begin{cases} {}^c D^\alpha x(t) = f(t, x(t), (Sx)(t)), t \in J = [0, T], \alpha \in (n-1, n), \\ x(0) = x_0, x'(0) = x_0^1, x''(0) = x_0^2, \dots, x^{(n-2)}(0) = x_0^{n-2}, \\ x^{(n-1)}(T) = x_T, \end{cases}$$

where ${}^c D^\alpha$ is the Caputo fractional derivative of order α , $f : J \times X \times X \rightarrow X$ is a given function satisfying some assumptions that will be specified later and $x_0, x_0^i (i = 1, 2, \dots, n-2, n \geq 3, n \text{ is an integer})$, x_T are elements of X , and S is a nonlinear integral operator given by $(Sx)(t) = \int_0^t k(t, s, x(s))ds$, where $k \in C(J \times J \times X, X)$.

We investigate existence and uniqueness results for the fractional boundary value problem (BVP for short), (1) by using fractional calculus, fixed point theorems and Hölder inequality. Compared the present paper with the work [4], there are at least three differences: (i) the operator $(Sx)(t)$ is not linear but nonlinear operator; (ii) another singular Gronwall's inequality is given by us (Lemma 3.2) to obtain the priori bounds; (iii) some new hypotheses applied on the function f . Our attempt is to generalize the results proved in [1, 4, 24].

This paper is organized as follows. In Section 2, we set forth some preliminaries. Section 3 introduces a new generalized singular type Gronwall inequality to establish the estimate for priori bounds. In Section 4, we prove our main results by applying Banach contraction principle and Schaefer's fixed point theorem. Finally, in Section 5, applications of the main results are exhibited.

2. PRELIMINARIES

In this section, we set forth some preliminaries from [10, 25].

Let X be a Banach space with the norm $\|\cdot\|$. We denote by $C(J, X)$ the space of X -valued continuous functions on J with the supremum norm $\|x\|_\infty := \sup\{\|x(t)\| : t \in J\}$. For measurable functions $m : J \rightarrow \mathbb{R}$, define the norm $\|m\|_{L^p(J, \mathbb{R})} = \left(\int_J |m(t)|^p dt\right)^{\frac{1}{p}}$, $1 \leq p < \infty$, where $L^p(J, \mathbb{R})$ the Banach space of all Lebesgue measurable functions m with $\|m\|_{L^p(J, \mathbb{R})} < \infty$.

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a suitable function h is defined by*

$$I_{a+}^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds,$$

where $a \in \mathbb{R}$ and Γ is the Gamma function.

Definition 2.2. *For a suitable function h given on the interval $[a, b]$, the Riemann-Liouville fractional derivative of order $\alpha > 0$ of h , is defined by*

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s) ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of α .

Definition 2.3. *For a suitable function h given on the interval $[a, b]$, the Caputo fractional order derivative of order $\alpha > 0$ of h , is defined by*

$$({}^c D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of α .

Remark 2.1. (i) *The Caputo derivative of a constant is equal to zero.*

(ii) *If h is an abstract function with values in X , then integrals which appeared in Definitions 2.1, 2.2 and 2.3 are taken in Bochner's sense.*

Lemma 2.1. ([25]) *Let $\alpha > 0$; then the differential equation ${}^c D^\alpha h(t) = 0$, has the following general solution $h(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}$, where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, where $n = [\alpha] + 1$.*

Lemma 2.2. ([25]) *Let $\alpha > 0$; then*

$$I^\alpha ({}^c D^\alpha h)(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1},$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, where $n = [\alpha] + 1$.

For more details, see [10].

Definition 2.4. A function $x \in C(J, X)$ with its α derivative existing on J is said to be a solution of the fractional BVP (1) if x satisfies the equation ${}^c D^\alpha x(t) = f(t, x(t), (Sx)(t))$ a.e. on J , and the conditions $x(0) = x_0, x'(0) = x_0^1, x''(0) = x_0^2, \dots, x^{(n-2)}(0) = x_0^{n-2}, x^{(n-1)}(T) = x_T$.

For the existence of solutions for the fractional BVP (1), we need the following auxiliary lemma.

Lemma 2.3. Let $\bar{f} : J \rightarrow X$ be continuous. A function $x \in C(J, X)$ is solution of the fractional integral equation

$$(2) \quad \begin{aligned} x(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bar{f}(s) ds \\ & - \frac{t^{n-1}}{(n-1)! \Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} \bar{f}(s) ds \\ & + x_0 + x_0^1 t + \frac{x_0^2}{2!} t^2 + \dots + \frac{x_0^{n-2}}{(n-2)!} t^{n-2} + \frac{x_T}{(n-1)!} t^{n-1}, \end{aligned}$$

if and only if x is a solution of the following fractional BVP

$$(3) \quad {}^c D^\alpha x(t) = \bar{f}(t), t \in J = [0, T], \alpha \in (n-1, n),$$

$$(4) \quad x(0) = x_0, x'(0) = x_0^1, x''(0) = x_0^2, \dots, x^{(n-2)}(0) = x_0^{n-2}, x^{(n-1)}(T) = x_T.$$

Proof Assume that x satisfies fractional BVP (3)-(4); then by using Lemma 2.2 and Def. 2.1, we get

$$x(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bar{f}(s) ds,$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, \dots, n-1$. That is:

$$(5) \quad x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bar{f}(s) ds - c_0 - c_1 t - c_2 t^2 - \dots - c_{n-2} t^{n-2} - c_{n-1} t^{n-1}.$$

By applying first condition of (4) i.e $x(0) = x_0$, we have

$$x_0 = -c_0 \Rightarrow c_0 = -x_0.$$

Now,

$$\begin{aligned} x'(t) = & \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \bar{f}(s) ds - c_1 - 2c_2 t - \dots \\ & - (n-2)c_{n-2} t^{n-3} - (n-1)c_{n-1} t^{n-2}, \end{aligned}$$

by using $x'(0) = x_0^1$, we get

$$x_0^1 = -c_1 \Rightarrow c_1 = -x_0^1.$$

And,

$$x''(t) = \frac{1}{\Gamma(\alpha - 2)} \int_0^t (t - s)^{\alpha-3} \bar{f}(s) ds - 2c_2 - \dots \\ - (n - 2)(n - 3)c_{n-2}t^{n-4} - (n - 1)(n - 2)c_{n-1}t^{n-3},$$

by using $x''(0) = x_0^2$, we have

$$x_0^2 = -2c_2 \Rightarrow c_2 = -\frac{x_0^2}{2!}.$$

By continuing this process, we have

$$x^{(n-2)}(t) = \frac{1}{\Gamma(\alpha - n + 2)} \int_0^t (t - s)^{\alpha-n+1} \bar{f}(s) ds \\ - (n - 2)(n - 3)(n - 4) \dots (2)(1)c_{n-2} \\ - (n - 1)(n - 2)(n - 3) \dots (3)(2)c_{n-1}t,$$

by using $x^{(n-2)}(0) = x_0^{n-2}$, we have

$$x_0^{n-2} = -(n - 2)!c_{n-2} \Rightarrow c_{n-2} = -\frac{x_0^{n-2}}{(n - 2)!}.$$

Finally,

$$x^{(n-1)}(t) = \frac{1}{\Gamma(\alpha - n + 1)} \int_0^t (t - s)^{\alpha-n} \bar{f}(s) ds \\ - (n - 1)(n - 2)(n - 3) \dots (3)(2)(1)c_{n-1} \\ = \frac{1}{\Gamma(\alpha - n + 1)} \int_0^t (t - s)^{\alpha-n} \bar{f}(s) ds - (n - 1)!c_{n-1},$$

by using $x^{(n-1)}(T) = x_T$, we obtain

$$x_T = \frac{1}{\Gamma(\alpha - n + 1)} \int_0^T (T - s)^{\alpha-n} \bar{f}(s) ds - (n - 1)!c_{n-1}.$$

Hence,

$$c_{n-1} = -\frac{x_T}{(n - 1)!} + \frac{1}{(n - 1)!\Gamma(\alpha - n + 1)} \int_0^T (T - s)^{\alpha-n} \bar{f}(s) ds.$$

Now, by substituting the values of $c_i (i = 0, 1, 2, \dots, n - 1)$ in (5), we obtain

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \bar{f}(s) ds \\ - \frac{t^{n-1}}{(n - 1)!\Gamma(\alpha - n + 1)} \int_0^T (T - s)^{\alpha-n} \bar{f}(s) ds \\ + x_0 + x_0^1 t + \frac{x_0^2}{2!} t^2 + \dots + \frac{x_0^{n-2}}{(n - 2)!} t^{n-2} + \frac{x_T}{(n - 1)!} t^{n-1}.$$

Conversely, assume that if x satisfies fractional integral equation (2), if $t \in [0, T]$ then $x(0) = x_0, x'(0) = x_0^1, x''(0) = x_0^2, \dots, x^{(n-2)}(0) = x_0^{n-2}, x^{(n-1)}(T) = x_T$ and applying Remark 2.1 (i)-(iii), we get (3) is also satisfied.

□

As a consequence of lemma 2.3, we have the following result which is useful in what follows.

Lemma 2.4. *Let $f : J \times X \times X \rightarrow X$ be continuous function. Then, $x \in C(J, X)$ is a solution of the fractional integral equation*

$$\begin{aligned} x(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), (Sx)(s)) ds \\ & - \frac{t^{n-1}}{(n-1)! \Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} f(s, x(s), (Sx)(s)) ds \\ & + x_0 + x_0^1 t + \frac{x_0^2}{2!} t^2 + \dots + \frac{x_0^{n-2}}{(n-2)!} t^{n-2} + \frac{x_T}{(n-1)!} t^{n-1}, \end{aligned}$$

if and only if x is solution of the fractional BVP (1).

Lemma 2.5. *(Bochner theorem) A measurable function $f : J \rightarrow X$ is Bochner integrable if $\|f\|$ is Lebesgue integrable.*

Lemma 2.6. *(Mazur theorem, [18]) Let X be a Banach space. If $U \subset X$ is relatively compact, then $\text{conv}(U)$ is relatively compact and $\overline{\text{conv}}(U)$ is compact.*

Lemma 2.7. *(Ascoli-Arzelà theorem) Let $S = \{s(t)\}$ is a function family of continuous mappings $s : [a, b] \rightarrow X$. If S is uniformly bounded and equicontinuous, and for any $t^* \in [a, b]$, the set $\{s(t^*)\}$ is relatively compact, then, there exists a uniformly convergent function sequence $\{s_n(t)\} (n = 1, 2, \dots, t \in [a, b])$ in S .*

Lemma 2.8. *(Schaefer's fixed point theorem) Let $F : X \rightarrow X$ be a completely continuous operator. If the set $E(F) = \{x \in X : x = \eta Fx \text{ for some } \eta \in [0, 1]\}$ is bounded, then, F has fixed points.*

3. A GENERALIZED SINGULAR TYPE GRONWALL'S INEQUALITY

In order to apply the Schaefer's fixed point theorem to establish the existence of solutions, we need to introduce a new generalized singular Gronwall type inequality with mixed type singular integral operator. It plays a fundamental role in the study of BVP for nonlinear differential equations of fractional order.

We, first, state a generalized Gronwall inequality from [21].

Lemma 3.1. *(Lemma 3.2, [21]) Let $x \in C(J, X)$ satisfies the following inequality:*

$$\|x(t)\| \leq a + b \int_0^t \|x(\theta)\|^{\lambda_1} d\theta + c \int_0^T \|x(\theta)\|^{\lambda_2} d\theta + d \int_0^t \|x_\theta\|_B^{\lambda_3} d\theta$$

$$+ e \int_0^T \|x_\theta\|_B^{\lambda_4} d\theta, t \in J,$$

where $\lambda_1, \lambda_3 \in [0, 1]$, $\lambda_2, \lambda_4 \in [0, 1)$, $a, b, c, d, e \geq 0$ are constants and $\|x_\theta\|_B = \sup_{0 \leq s \leq \theta} \|x(s)\|$. Then there exists a constant $L > 0$ such that

$$\|x(t)\| \leq L.$$

Using the above generalized Gronwall inequality, we can obtain the following new generalized singular type Gronwall inequality.

Lemma 3.2. *Let $x \in C(J, X)$ satisfies the following inequality:*

$$\begin{aligned} \|x(t)\| \leq & a + b \int_0^t (t-s)^{\alpha-1} \|x(s)\|^\lambda ds + c \int_0^T (T-s)^{\alpha-n} \|x(s)\|^\lambda ds \\ & + d \int_0^t (t-s)^{\alpha-1} \|x_s\|_B^\lambda ds + e \int_0^T (T-s)^{\alpha-n} \|x_s\|_B^\lambda ds, \end{aligned} \quad (6)$$

where $\alpha \in (n-1, n)$, $\lambda \in [0, 1 - \frac{1}{p})$ for some $1 < p < \frac{1}{n-\alpha}$, $\|x_s\|_B = \sup_{0 \leq \tau \leq s} \|x(\tau)\|$ and $a, b, c, d, e \geq 0$ are constants. Then, there exists a constant $L > 0$, such that

$$\|x(t)\| \leq L.$$

Proof Let

$$y(t) = \begin{cases} 1, & \|x(t)\| \leq 1, \\ x(t), & \|x(t)\| > 1. \end{cases}$$

Using (6) and Hölder inequality, we get

$$\begin{aligned} \|x(t)\| & \leq \|y(t)\| \\ & \leq (a+1) + b \int_0^t (t-s)^{\alpha-1} \|y(s)\|^\lambda ds + c \int_0^T (T-s)^{\alpha-n} \|y(s)\|^\lambda ds \\ & \quad + d \int_0^t (t-s)^{\alpha-1} \|y_s\|_B^\lambda ds + e \int_0^T (T-s)^{\alpha-n} \|y_s\|_B^\lambda ds \\ & \leq (a+1) + b \left(\int_0^t (t-s)^{p(\alpha-1)} ds \right)^{\frac{1}{p}} \left(\int_0^t \|y(s)\|^{\frac{\lambda p}{p-1}} ds \right)^{\frac{p-1}{p}} \\ & \quad + c \left(\int_0^T (T-s)^{p(\alpha-n)} ds \right)^{\frac{1}{p}} \left(\int_0^T \|y(s)\|^{\frac{\lambda p}{p-1}} ds \right)^{\frac{p-1}{p}} \\ & \quad + d \|y_s\|_B^\lambda \int_0^t (t-s)^{\alpha-1} ds + e \|y_s\|_B^\lambda \int_0^T (T-s)^{\alpha-n} ds \\ & \leq (a+1) + b \left(\frac{t^{p(\alpha-1)+1}}{p(\alpha-1)+1} \right)^{\frac{1}{p}} \int_0^t \|y(s)\|^{\frac{\lambda p}{p-1}} ds \end{aligned}$$

$$\begin{aligned}
& + c \left(\frac{T^{p(\alpha-n)+1}}{p(\alpha-n)+1} \right)^{\frac{1}{p}} \int_0^T \|y(s)\|_{\frac{\lambda p}{p-1}} ds \\
& + d \|y_s\|_B^\lambda \frac{t^\alpha}{\alpha} + e \|y_s\|_B^\lambda \frac{T^{\alpha-n+1}}{(\alpha-n+1)} \\
\leq & (a+1) + d \|y_s\|_B^\lambda \frac{T^\alpha}{\alpha} + e \|y_s\|_B^\lambda \frac{T^{\alpha-n+1}}{(\alpha-n+1)} \\
& + b \left(\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1} \right)^{\frac{1}{p}} \int_0^t \|y(s)\|_{\frac{\lambda p}{p-1}} ds \\
& + c \left(\frac{T^{p(\alpha-n)+1}}{p(\alpha-n)+1} \right)^{\frac{1}{p}} \int_0^T \|y(s)\|_{\frac{\lambda p}{p-1}} ds,
\end{aligned}$$

where $0 < \frac{\lambda p}{p-1} < 1$.

Hence, by lemma 3.1 there exists a constant $L > 0$, such that $\|x(t)\| \leq L$.

□

4. MAIN RESULTS

For convenience, we list hypotheses that will be used in our further discussion.

- (H1) The function $f : J \times X \times X \rightarrow X$ is measurable with respect to t on J and is continuous with respect to x on X .
- (H2) There exists a constant $\alpha_1 \in (0, \alpha - n + 1)$ and real-valued functions $m_1(t), m_2(t) \in L^{\frac{1}{\alpha_1}}(J, \mathbb{R})$, such that

$$\begin{aligned}
\|f(t, x(t), (Sx)(t)) - f(t, y(t), (Sy)(t))\| & \leq m_1(t)(\|x(t) - y(t)\| + \|Sx(t) - Sy(t)\|), \\
\|k(t, s, x(s)) - k(t, s, y(s))\| & \leq m_2(t)\|x(s) - y(s)\|,
\end{aligned}$$

for each $s \in [0, t]$, $t \in J$ and all $x, y \in X$.

- (H3) There exists a constant $\alpha_2 \in (0, \alpha - n + 1)$ and real-valued function $h(t) \in L^{\frac{1}{\alpha_2}}(J, \mathbb{R})$, such that $\|f(t, x(t), (Sx)(t))\| \leq h(t)$, for each $t \in J$, and all $x \in X$.

For brevity, let $M = \|m_1 + m_1 m_2 T\|_{L^{\frac{1}{\alpha_1}}(J, \mathbb{R})}$ and $H = \|h\|_{L^{\frac{1}{\alpha_2}}(J, \mathbb{R})}$.

- (H4) There exist constants $\lambda \in [0, 1 - \frac{1}{p}]$ for some $1 < p < \frac{1}{n-\alpha}$ and $N_f, N_k > 0$, such that

$$\begin{aligned}
\|f(t, x(t), (Sx)(t))\| & \leq N_f(1 + \|x(t)\|^\lambda + \|(Sx)(t)\|), \\
\|k(t, s, x(s))\| & \leq N_k(1 + \|x(s)\|^\lambda),
\end{aligned}$$

for each $s \in [0, t]$, $t \in J$ and all $x \in X$.

- (H5) For every $t \in J$, the sets $K_1 = \{(t-s)^{\alpha-1} f(s, x(s), (Sx)(s)) : x \in C(J, X), s \in [0, t]\}$ and

$K_2 = \{(t-s)^{\alpha-n} f(s, x(s), (Sx)(s)) : x \in C(J, X), s \in [0, t]\}$ are relatively compact.

Now, we are in position to deal with our main results.

Theorem 4.1. *Assume that (H1)-(H3) hold. If*

$$(7) \quad \Omega_{\alpha, T, n} = \frac{M}{\Gamma(\alpha)} \frac{T^{\alpha-\alpha_1}}{\left(\frac{\alpha-\alpha_1}{1-\alpha_1}\right)^{1-\alpha_1}} + \frac{M}{(n-1)! \Gamma(\alpha-n+1)} \frac{T^{\alpha-\alpha_1}}{\left(\frac{\alpha-\alpha_1-n+1}{1-\alpha_1}\right)^{1-\alpha_1}} < 1.$$

Then, the fractional BVP (1) has a unique solution on J .

Proof By making use of hypothesis (H3) and Hölder inequality, for each $t \in J$, we have

$$\begin{aligned} \int_0^t \|(t-s)^{\alpha-1} f(s, x(s), (Sx)(s))\| ds &\leq \int_0^t (t-s)^{\alpha-1} \|f(s, x(s), (Sx)(s))\| ds \\ &\leq \int_0^t (t-s)^{\alpha-1} h(s) ds \\ &\leq \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_2}} ds \right)^{1-\alpha_2} \left(\int_0^t (h(s))^{\frac{1}{\alpha_2}} ds \right)^{\alpha_2} \\ &\leq H \left(\left[\frac{-(t-s)^{\frac{\alpha-1}{1-\alpha_2}+1}}{\frac{\alpha-1}{1-\alpha_2}+1} \right]_0^t \right)^{1-\alpha_2} \leq H \frac{T^{\alpha-\alpha_2}}{\left(\frac{\alpha-\alpha_2}{1-\alpha_2}\right)^{1-\alpha_2}}. \end{aligned}$$

Thus, $\|(t-s)^{\alpha-1} f(s, x(s), (Sx)(s))\|$ is Lebesgue integrable with respect to $s \in [0, t]$ for all $t \in J$ and $x \in C(J, X)$. Then, $(t-s)^{\alpha-1} f(s, x(s), (Sx)(s))$ is Bochner integrable with respect to $s \in [0, t]$ for all $t \in J$ due to lemma 2.5, and

$$\begin{aligned} \int_0^T \|(T-s)^{\alpha-n} f(s, x(s), (Sx)(s))\| ds &\leq \int_0^T (T-s)^{\alpha-n} \|f(s, x(s), (Sx)(s))\| ds \\ &\leq \int_0^T (T-s)^{\alpha-n} h(s) ds \\ &\leq \left(\int_0^T (T-s)^{\frac{\alpha-n}{1-\alpha_2}} ds \right)^{1-\alpha_2} \left(\int_0^T (h(s))^{\frac{1}{\alpha_2}} ds \right)^{\alpha_2} \\ &\leq H \left(\frac{T^{\frac{\alpha-\alpha_2-n+1}{1-\alpha_2}}}{\frac{\alpha-\alpha_2-n+1}{1-\alpha_2}} \right)^{1-\alpha_2} \leq H \frac{T^{\alpha-\alpha_2-n+1}}{\left(\frac{\alpha-\alpha_2-n+1}{1-\alpha_2}\right)^{1-\alpha_2}}. \end{aligned}$$

Thus, $\|(T-s)^{\alpha-n} f(s, x(s), (Sx)(s))\|$ is Lebesgue integrable with respect to $s \in [0, T]$ and $x \in C(J, X)$. Then, $(T-s)^{\alpha-n} f(s, x(s), (Sx)(s))$ is Bochner integrable with respect to $s \in [0, T]$ due to lemma 2.5.

Hence, the fractional BVP (1) is equivalent to the following fractional integral equation

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), (Sx)(s)) ds$$

$$\begin{aligned}
& - \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} f(s, x(s), (Sx)(s)) ds \\
& + x_0 + x_0^1 t + \frac{x_0^2}{2!} t^2 + \dots + \frac{x_0^{n-2}}{(n-2)!} t^{n-2} + \frac{x_T}{(n-1)!} t^{n-1}.
\end{aligned}$$

Now, let $B_r = \{x \in C(J, X) : \|x\|_\infty \leq r\}$, where

$$\begin{aligned}
(8) \quad r \geq & \frac{HT^{\alpha-\alpha_2}}{\Gamma(\alpha)\left(\frac{\alpha-\alpha_2}{1-\alpha_2}\right)^{1-\alpha_2}} + \frac{HT^{\alpha-\alpha_2}}{(n-1)!\Gamma(\alpha-n+1)} \left(\frac{\alpha-\alpha_2-n+1}{1-\alpha_2}\right)^{-(1-\alpha_2)} \\
& + \|x_0\| + \|x_0^1\|T + \frac{\|x_0^2\|}{2!} T^2 + \dots + \frac{\|x_0^{n-2}\|}{(n-2)!} T^{n-2} + \frac{\|x_T\|}{(n-1)!} T^{n-1}.
\end{aligned}$$

Define the operator F on B_r as follows:

$$\begin{aligned}
(9) \quad (F(x))(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), (Sx)(s)) ds \\
& - \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} f(s, x(s), (Sx)(s)) ds \\
& + x_0 + x_0^1 t + \frac{x_0^2}{2!} t^2 + \dots + \frac{x_0^{n-2}}{(n-2)!} t^{n-2} + \frac{x_T}{(n-1)!} t^{n-1}, t \in J.
\end{aligned}$$

Clearly, the solution of the fractional BVP (1) is the fixed point of the operator F on B_r . We shall use the Banach contraction principle to prove that F has a fixed point. The proof is divided into two steps.

Step 1. $F(x) \in B_r$ for every $x \in B_r$.

For every $x \in B_r$ and $\delta > 0$, by (H3) and Hölder inequality, we have

$$\begin{aligned}
& \|(F(x))(t+\delta) - (F(x))(t)\| \\
& \leq \left\| \frac{1}{\Gamma(\alpha)} \int_0^{t+\delta} (t+\delta-s)^{\alpha-1} f(s, x(s), (Sx)(s)) ds \right. \\
& \quad \left. - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), (Sx)(s)) ds \right\| \\
& \quad + \left\| \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} f(s, x(s), (Sx)(s)) ds \right. \\
& \quad \left. - \frac{(t+\delta)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} f(s, x(s), (Sx)(s)) ds \right\| \\
& \quad + \left\| x_0^1(t+\delta-t) + \frac{x_0^2}{2!} [(t+\delta)^2 - t^2] + \dots \right. \\
& \quad \left. + \frac{x_0^{n-2}}{(n-2)!} [(t+\delta)^{n-2} - t^{n-2}] + \frac{x_T}{(n-1)!} [(t+\delta)^{n-1} - t^{n-1}] \right\|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t [(t+\delta-s)^{\alpha-1} - (t-s)^{\alpha-1}] \|f(s, x(s), (Sx)(s))\| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_t^{t+\delta} (t+\delta-s)^{\alpha-1} \|f(s, x(s), (Sx)(s))\| ds \\
&\quad + \left[\frac{(t+\delta)^{n-1} - t^{n-1}}{(n-1)! \Gamma(\alpha-n+1)} \right] \int_0^T (T-s)^{\alpha-n} \|f(s, x(s), (Sx)(s))\| ds \\
&\quad + \|x_0^1\| (t+\delta-t) + \frac{\|x_0^2\|}{2!} [(t+\delta)^2 - t^2] + \dots \\
&\quad + \frac{\|x_0^{n-2}\|}{(n-2)!} [(t+\delta)^{n-2} - t^{n-2}] + \frac{\|x_T\|}{(n-1)!} [(t+\delta)^{n-1} - t^{n-1}] \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t [(t+\delta-s)^{\alpha-1} - (t-s)^{\alpha-1}] h(s) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_t^{t+\delta} (t+\delta-s)^{\alpha-1} h(s) ds \\
&\quad + \left[\frac{(t+\delta)^{n-1} - t^{n-1}}{(n-1)! \Gamma(\alpha-n+1)} \right] \int_0^T (T-s)^{\alpha-n} h(s) ds \\
&\quad + \|x_0^1\| (t+\delta-t) + \frac{\|x_0^2\|}{2!} [(t+\delta)^2 - t^2] + \dots \\
&\quad + \frac{\|x_0^{n-2}\|}{(n-2)!} [(t+\delta)^{n-2} - t^{n-2}] + \frac{\|x_T\|}{(n-1)!} [(t+\delta)^{n-1} - t^{n-1}] \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\int_0^t [(t+\delta-s)^{\alpha-1} - (t-s)^{\alpha-1}]^{\frac{1}{1-\alpha_2}} ds \right)^{1-\alpha_2} \left(\int_0^t (h(s))^{\frac{1}{\alpha_2}} ds \right)^{\alpha_2} \\
&\quad + \frac{1}{\Gamma(\alpha)} \left(\int_t^{t+\delta} (t+\delta-s)^{\frac{\alpha-1}{1-\alpha_2}} ds \right)^{1-\alpha_2} \left(\int_t^{t+\delta} (h(s))^{\frac{1}{\alpha_2}} ds \right)^{\alpha_2} \\
&\quad + \left[\frac{(t+\delta)^{n-1} - t^{n-1}}{(n-1)! \Gamma(\alpha-n+1)} \right] \left(\int_0^T (T-s)^{\frac{\alpha-n}{1-\alpha_2}} ds \right)^{1-\alpha_2} \left(\int_0^T (h(s))^{\frac{1}{\alpha_2}} ds \right)^{\alpha_2} \\
&\quad + \|x_0^1\| (t+\delta-t) + \frac{\|x_0^2\|}{2!} [(t+\delta)^2 - t^2] + \dots \\
&\quad + \frac{\|x_0^{n-2}\|}{(n-2)!} [(t+\delta)^{n-2} - t^{n-2}] + \frac{\|x_T\|}{(n-1)!} [(t+\delta)^{n-1} - t^{n-1}] \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\int_0^t [(t+\delta-s)^{\frac{\alpha-1}{1-\alpha_2}} - (t-s)^{\frac{\alpha-1}{1-\alpha_2}}] ds \right)^{1-\alpha_2} \left(\int_0^t (h(s))^{\frac{1}{\alpha_2}} ds \right)^{\alpha_2} \\
&\quad + \frac{1}{\Gamma(\alpha)} \left(\int_t^{t+\delta} (t+\delta-s)^{\frac{\alpha-1}{1-\alpha_2}} ds \right)^{1-\alpha_2} \left(\int_t^{t+\delta} (h(s))^{\frac{1}{\alpha_2}} ds \right)^{\alpha_2} \\
&\quad + \left[\frac{(t+\delta)^{n-1} - t^{n-1}}{(n-1)! \Gamma(\alpha-n+1)} \right] \left(\int_0^T (T-s)^{\frac{\alpha-n}{1-\alpha_2}} ds \right)^{1-\alpha_2} \left(\int_0^T (h(s))^{\frac{1}{\alpha_2}} ds \right)^{\alpha_2}
\end{aligned}$$

$$\begin{aligned}
& + \|x_0^1\|[(t + \delta - t) + \frac{\|x_0^2\|}{2!}[(t + \delta)^2 - t^2] + \dots \\
& + \frac{\|x_0^{n-2}\|}{(n-2)!}[(t + \delta)^{n-2} - t^{n-2}] + \frac{\|x_T\|}{(n-1)!}[(t + \delta)^{n-1} - t^{n-1}] \\
\leq & \frac{H}{\Gamma(\alpha)} \left(\frac{-\delta^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}} + \frac{(t + \delta)^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}} - \frac{t^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}} \right)^{1-\alpha_2} + \frac{H}{\Gamma(\alpha)} \left(\frac{\delta^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}} \right)^{1-\alpha_2} \\
& + \left[\frac{(t + \delta)^{n-1} - t^{n-1}}{(n-1)!\Gamma(\alpha - n + 1)} \right] H \left(\frac{T^{\frac{\alpha-\alpha_2-n+1}{1-\alpha_2}}}{\frac{\alpha-\alpha_2-n+1}{1-\alpha_2}} \right)^{1-\alpha_2} \\
& + \|x_0^1\|[(t + \delta - t) + \frac{\|x_0^2\|}{2!}[(t + \delta)^2 - t^2] + \dots \\
& + \frac{\|x_0^{n-2}\|}{(n-2)!}[(t + \delta)^{n-2} - t^{n-2}] + \frac{\|x_T\|}{(n-1)!}[(t + \delta)^{n-1} - t^{n-1}].
\end{aligned}$$

It is obvious that the right-hand side of the above inequality tends to zero as $\delta \rightarrow 0$. Therefore, F is continuous on J , that is, $F(x) \in C(J, X)$. Moreover, for $x \in B_r$ and all $t \in J$, by using (8), we have

$$\begin{aligned}
\|(F(x))(t)\| & \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x(s), (Sx)(s))\| ds \\
& + \frac{t^{n-1}}{(n-1)!\Gamma(\alpha - n + 1)} \int_0^T (T-s)^{\alpha-n} \|f(s, x(s), (Sx)(s))\| ds \\
& + \|x_0\| + \|x_0^1\|t + \frac{\|x_0^2\|}{2!}t^2 + \dots + \frac{\|x_0^{n-2}\|}{(n-2)!}t^{n-2} + \frac{\|x_T\|}{(n-1)!}t^{n-1} \\
\leq & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \\
& + \frac{t^{n-1}}{(n-1)!\Gamma(\alpha - n + 1)} \int_0^T (T-s)^{\alpha-n} h(s) ds \\
& + \|x_0\| + \|x_0^1\|t + \frac{\|x_0^2\|}{2!}t^2 + \dots + \frac{\|x_0^{n-2}\|}{(n-2)!}t^{n-2} + \frac{\|x_T\|}{(n-1)!}t^{n-1} \\
\leq & \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_2}} ds \right)^{1-\alpha_2} \left(\int_0^t (h(s))^{\frac{1}{\alpha_2}} ds \right)^{\alpha_2} \\
& + \frac{t^{n-1}}{(n-1)!\Gamma(\alpha - n + 1)} \left(\int_0^T (T-s)^{\frac{\alpha-n}{1-\alpha_2}} ds \right)^{1-\alpha_2} \left(\int_0^T (h(s))^{\frac{1}{\alpha_2}} ds \right)^{\alpha_2} \\
& + \|x_0\| + \|x_0^1\|t + \frac{\|x_0^2\|}{2!}t^2 + \dots + \frac{\|x_0^{n-2}\|}{(n-2)!}t^{n-2} + \frac{\|x_T\|}{(n-1)!}t^{n-1}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{HT^{\alpha-\alpha_2}}{\Gamma(\alpha)\left(\frac{\alpha-\alpha_2}{1-\alpha_2}\right)^{1-\alpha_2}} + \frac{HT^{\alpha-\alpha_2}}{(n-1)!\Gamma(\alpha-n+1)} \left(\frac{\alpha-\alpha_2-n+1}{1-\alpha_2}\right)^{-(1-\alpha_2)} \\
&\quad + \|x_0\| + \|x_0^1\|T + \frac{\|x_0^2\|}{2!}T^2 + \dots + \frac{\|x_0^{n-2}\|}{(n-2)!}T^{n-2} + \frac{\|x_T\|}{(n-1)!}T^{n-1} \\
&\leq r.
\end{aligned}$$

Thus, $\|F(x)\|_\infty \leq r$ and we conclude that for all $x \in B_r$, $F(x) \in B_r$, that is, $F : B_r \rightarrow B_r$.

Step 2. F is contraction mapping on B_r .

For $x, y \in B_r$ and any $t \in J$, by using (7), (H2) and Hölder inequality, we have

$$\begin{aligned}
&\|(F(x))(t) - (F(y))(t)\| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x(s), (Sx)(s)) - f(s, y(s), (Sy)(s))\| ds \\
&\quad + \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} \|f(s, x(s), (Sx)(s)) - f(s, y(s), (Sy)(s))\| ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m_1(s) (\|x(s) - y(s)\| + \|(Sx)(s) - (Sy)(s)\|) ds \\
&\quad + \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} m_1(s) (\|x(s) - y(s)\| + \|(Sx)(s) - (Sy)(s)\|) ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m_1(s) \\
&\quad \times \left(\|x(s) - y(s)\| + \int_0^s \|k(s, \tau, x(\tau)) - k(s, \tau, y(\tau))\| d\tau \right) ds \\
&\quad + \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} m_1(s) \\
&\quad \times \left(\|x(s) - y(s)\| + \int_0^s \|k(s, \tau, x(\tau)) - k(s, \tau, y(\tau))\| d\tau \right) ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m_1(s) \\
&\quad \times \left(\|x(s) - y(s)\| + \int_0^s m_2(s) \|x(\tau) - y(\tau)\| d\tau \right) ds \\
&\quad + \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} m_1(s) \\
&\quad \times \left(\|x(s) - y(s)\| + \int_0^s m_2(s) \|x(\tau) - y(\tau)\| d\tau \right) ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m_1(s) (\|x - y\|_\infty + m_2(s)T\|x - y\|_\infty) ds
\end{aligned}$$

$$\begin{aligned}
& + \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} m_1(s) (\|x-y\|_\infty + m_2(s)T\|x-y\|_\infty) ds \\
& \leq \frac{\|x-y\|_\infty}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}} ds \right)^{1-\alpha_1} \left(\int_0^t (m_1(s) + m_1(s)m_2(s)T)^{\frac{1}{\alpha_1}} ds \right)^{\alpha_1} \\
& + \frac{t^{n-1}\|x-y\|_\infty}{(n-1)!\Gamma(\alpha-n+1)} \left(\int_0^T (T-s)^{\frac{\alpha-n}{1-\alpha_1}} ds \right)^{1-\alpha_1} \\
& \times \left(\int_0^T (m_1(s) + m_1(s)m_2(s)T)^{\frac{1}{\alpha_1}} ds \right)^{\alpha_1} \\
& \leq \frac{M\|x-y\|_\infty}{\Gamma(\alpha)} \left(\frac{t^{\frac{\alpha-\alpha_1}{1-\alpha_1}}}{\frac{\alpha-\alpha_1}{1-\alpha_1}} \right)^{1-\alpha_1} + \frac{M\|x-y\|_\infty t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \left(\frac{T^{\frac{\alpha-\alpha_1-n+1}{1-\alpha_1}}}{\frac{\alpha-\alpha_1-n+1}{1-\alpha_1}} \right)^{1-\alpha_1} \\
& \leq \left(\frac{M}{\Gamma(\alpha)} \frac{T^{\alpha-\alpha_1}}{\left(\frac{\alpha-\alpha_1}{1-\alpha_1}\right)^{1-\alpha_1}} + \frac{M}{(n-1)!\Gamma(\alpha-n+1)} \frac{T^{\alpha-\alpha_1}}{\left(\frac{\alpha-\alpha_1-n+1}{1-\alpha_1}\right)^{1-\alpha_1}} \right) \|x-y\|_\infty. \\
& = \Omega_{\alpha,T,n} \|x-y\|_\infty.
\end{aligned}$$

Thus, we have

$$\|F(x) - F(y)\|_\infty \leq \Omega_{\alpha,T,n} \|x - y\|_\infty.$$

Since $\Omega_{\alpha,T,n} < 1$, F is contraction. By Banach contraction principle, we can deduce that F has a unique fixed point which is the unique solution of the fractional BVP (1). \square

Our second main result is based on the well known Schaefer's fixed point theorem.

Theorem 4.2. *Assume that (H1), (H4) and (H5) hold. Then the fractional BVP (1) has at least one solution on J .*

Proof Transform the fractional BVP (1) into a fixed point problem. Consider the operator $F : C(J, X) \rightarrow C(J, X)$ defined as (9). It is obvious that F is well defined due to (H1), Hölder inequality and the lemma 2.5.

For the sake of convenience, we subdivide the proof into several steps.

Step 1. F is continuous operator.

Let $\{x_n\}$ be a sequence such that $x_n \rightarrow x$ in $C(J, X)$. Then for each $t \in J$, we have

$$\begin{aligned}
& \|(F(x_n))(t) - (F(x))(t)\| \\
& \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x_n(s), (Sx_n)(s)) - f(s, x(s), (Sx)(s))\| ds \\
& + \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} \|f(s, x_n(s), (Sx_n)(s)) - f(s, x(s), (Sx)(s))\| ds
\end{aligned}$$

$$\begin{aligned}
&\leq \|f(\cdot, x_n(\cdot), (Sx_n)(\cdot)) - f(\cdot, x(\cdot), (Sx)(\cdot))\|_\infty \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\
&\quad + \frac{\|f(\cdot, x_n(\cdot), (Sx_n)(\cdot)) - f(\cdot, x(\cdot), (Sx)(\cdot))\|_\infty t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} ds \\
&\leq \|f(\cdot, x_n(\cdot), (Sx_n)(\cdot)) - f(\cdot, x(\cdot), (Sx)(\cdot))\|_\infty \frac{t^\alpha}{\alpha\Gamma(\alpha)} \\
&\quad + \frac{\|f(\cdot, x_n(\cdot), (Sx_n)(\cdot)) - f(\cdot, x(\cdot), (Sx)(\cdot))\|_\infty t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \frac{T^{\alpha-n+1}}{(\alpha-n+1)} \\
&\leq \left(\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T^\alpha}{(n-1)!\Gamma(\alpha-n+2)} \right) \|f(\cdot, x_n(\cdot), (Sx_n)(\cdot)) - f(\cdot, x(\cdot), (Sx)(\cdot))\|_\infty.
\end{aligned}$$

Taking supremum, we get

$$\begin{aligned}
&\|Fx_n - Fx\|_\infty \\
&\leq \left(\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T^\alpha}{(n-1)!\Gamma(\alpha-n+2)} \right) \|f(\cdot, x_n(\cdot), (Sx_n)(\cdot)) - f(\cdot, x(\cdot), (Sx)(\cdot))\|_\infty,
\end{aligned}$$

since f is continuous, we have

$$\|Fx_n - Fx\|_\infty \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore, F is continuous operator.

Step 2. F maps bounded sets into bounded sets in $C(J, X)$.

Indeed, it is enough to show that for any $\eta^* > 0$, there exists a $l > 0$ such that for each $x \in B_{\eta^*} = \{x \in C(J, X) : \|x\|_\infty \leq \eta^*\}$, we have $\|Fx\|_\infty \leq l$.

For each $t \in J$, by (H4), we get

$$\begin{aligned}
\|(F(x))(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} N_f(1 + \|x(s)\|^\lambda + \|(Sx)(s)\|) ds \\
&\quad + \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} N_f(1 + \|x(s)\|^\lambda + \|(Sx)(s)\|) ds \\
&\quad + \|x_0\| + \|x_0^1\|t + \frac{\|x_0^2\|}{2!}t^2 + \dots + \frac{\|x_0^{n-2}\|}{(n-2)!}t^{n-2} + \frac{\|x_T\|}{(n-1)!}t^{n-1} \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} N_f \left(1 + \|x(s)\|^\lambda + \int_0^s \|k(s, \tau, x(\tau))\| d\tau \right) ds \\
&\quad + \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n}
\end{aligned}$$

$$\begin{aligned}
& \times N_f \left(1 + \|x(s)\|^\lambda + \int_0^s \|k(s, \tau, x(\tau))\| d\tau \right) ds \\
& + \|x_0\| + \|x_0^1\|t + \frac{\|x_0^2\|}{2!}t^2 + \dots + \frac{\|x_0^{n-2}\|}{(n-2)!}t^{n-2} + \frac{\|x_T\|}{(n-1)!}t^{n-1} \\
\leq & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} N_f \left(1 + \|x(s)\|^\lambda + \int_0^s N_k(1 + \|x(\tau)\|^\lambda) d\tau \right) ds \\
& + \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} \\
& \times N_f \left(1 + \|x(s)\|^\lambda + \int_0^s N_k(1 + \|x(\tau)\|^\lambda) d\tau \right) ds \\
& + \|x_0\| + \|x_0^1\|t + \frac{\|x_0^2\|}{2!}t^2 + \dots + \frac{\|x_0^{n-2}\|}{(n-2)!}t^{n-2} + \frac{\|x_T\|}{(n-1)!}t^{n-1} \\
\leq & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} N_f(1 + \|x\|_\infty^\lambda + N_k(1 + \|x\|_\infty^\lambda)T) ds \\
& + \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} \\
& \times N_f(1 + \|x\|_\infty^\lambda + N_k(1 + \|x\|_\infty^\lambda)T) ds \\
& + \|x_0\| + \|x_0^1\|t + \frac{\|x_0^2\|}{2!}t^2 + \dots + \frac{\|x_0^{n-2}\|}{(n-2)!}t^{n-2} + \frac{\|x_T\|}{(n-1)!}t^{n-1} \\
\leq & \frac{N_f(1 + (\eta^*)^\lambda)(1 + N_kT)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\
& + \frac{t^{n-1}N_f(1 + (\eta^*)^\lambda)(1 + N_kT)}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} ds \\
& + \|x_0\| + \|x_0^1\|t + \frac{\|x_0^2\|}{2!}t^2 + \dots + \frac{\|x_0^{n-2}\|}{(n-2)!}t^{n-2} + \frac{\|x_T\|}{(n-1)!}t^{n-1} \\
\leq & \frac{N_f(1 + (\eta^*)^\lambda)(1 + N_kT)}{\Gamma(\alpha)} \frac{t^\alpha}{\alpha} \\
& + \frac{t^{n-1}N_f(1 + (\eta^*)^\lambda)(1 + N_kT)}{(n-1)!\Gamma(\alpha-n+1)} \frac{T^{\alpha-n+1}}{(\alpha-n+1)} \\
& + \|x_0\| + \|x_0^1\|t + \frac{\|x_0^2\|}{2!}t^2 + \dots + \frac{\|x_0^{n-2}\|}{(n-2)!}t^{n-2} + \frac{\|x_T\|}{(n-1)!}t^{n-1} \\
\leq & \left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{(n-1)!\Gamma(\alpha-n+2)} \right) N_f(1 + (\eta^*)^\lambda)(1 + N_kT)T^\alpha \\
& + \|x_0\| + \|x_0^1\|T + \frac{\|x_0^2\|}{2!}T^2 + \dots + \frac{\|x_0^{n-2}\|}{(n-2)!}T^{n-2} + \frac{\|x_T\|}{(n-1)!}T^{n-1} \\
\leq & l,
\end{aligned}$$

where

$$l := \left(\frac{1}{\Gamma(\alpha + 1)} + \frac{1}{(n-1)!\Gamma(\alpha - n + 2)} \right) N_f(1 + (\eta^*)^\lambda)(1 + N_k T)T^\alpha \\ + \|x_0\| + \|x_0^1\|T + \frac{\|x_0^2\|}{2!}T^2 + \dots + \frac{\|x_0^{n-2}\|}{(n-2)!}T^{n-2} + \frac{\|x_T\|}{(n-1)!}T^{n-1}.$$

Thus, we have

$$\|(F(x))(t)\| \leq l \text{ and hence } \|Fx\|_\infty \leq l.$$

Step 3. F maps bounded sets into equicontinuous sets of $C(J, X)$.

Let $0 \leq t_1 \leq t_2 \leq T, x \in B_{\eta^*}$. Using (H4), again we have

$$\begin{aligned} & \|(F(x))(t_2) - (F(x))(t_1)\| \\ & \leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] N_f(1 + \|x\|_\infty^\lambda + N_k(1 + \|x\|_\infty^\lambda)T) ds \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} N_f(1 + \|x\|_\infty^\lambda + N_k(1 + \|x\|_\infty^\lambda)T) ds \\ & \quad + \frac{t_2^{n-1} - t_1^{n-1}}{(n-1)!\Gamma(\alpha - n + 1)} \int_0^T (T - s)^{\alpha-n} N_f(1 + \|x\|_\infty^\lambda + N_k(1 + \|x\|_\infty^\lambda)T) ds \\ & \quad + \|x_0^1\|(t_2 - t_1) + \frac{\|x_0^2\|}{2!}(t_2^2 - t_1^2) + \dots + \frac{\|x_0^{n-2}\|}{(n-2)!}(t_2^{n-2} - t_1^{n-2}) + \frac{\|x_T\|}{(n-1)!}(t_2^{n-1} - t_1^{n-1}) \\ & \leq \frac{N_f(1 + (\eta^*)^\lambda)(1 + N_k T)}{\Gamma(\alpha)} \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] ds \\ & \quad + \frac{N_f(1 + (\eta^*)^\lambda)(1 + N_k T)}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \\ & \quad + \frac{(t_2^{n-1} - t_1^{n-1})N_f(1 + (\eta^*)^\lambda)(1 + N_k T)}{(n-1)!\Gamma(\alpha - n + 1)} \int_0^T (T - s)^{\alpha-n} ds \\ & \quad + \|x_0^1\|(t_2 - t_1) + \frac{\|x_0^2\|}{2!}(t_2^2 - t_1^2) + \dots + \frac{\|x_0^{n-2}\|}{(n-2)!}(t_2^{n-2} - t_1^{n-2}) + \frac{\|x_T\|}{(n-1)!}(t_2^{n-1} - t_1^{n-1}) \\ & \leq \frac{N_f(1 + (\eta^*)^\lambda)(1 + N_k T)}{\Gamma(\alpha + 1)} (t_2^\alpha - t_1^\alpha) + \frac{(t_2^{n-1} - t_1^{n-1})N_f(1 + (\eta^*)^\lambda)(1 + N_k T)}{(n-1)!\Gamma(\alpha - n + 2)} T^{\alpha-n+1} \\ & \quad + \|x_0^1\|(t_2 - t_1) + \frac{\|x_0^2\|}{2!}(t_2^2 - t_1^2) + \dots + \frac{\|x_0^{n-2}\|}{(n-2)!}(t_2^{n-2} - t_1^{n-2}) + \frac{\|x_T\|}{(n-1)!}(t_2^{n-1} - t_1^{n-1}). \end{aligned}$$

As $t_2 \rightarrow t_1$, the right-hand side of the above inequality tends to zero and since x is an arbitrary in B_{η^*} , F is equicontinuous.

Now, let $\{x_n\}, n = 1, 2, \dots$ be a sequence on B_{η^*} , and

$$(Fx_n)(t) = (F_1x_n)(t) + (F_2x_n)(t) + (F_3x)(t), t \in J,$$

where

$$\begin{aligned} (F_1x_n)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_n(s), (Sx_n)(s)) ds, t \in J, \\ (F_2x_n)(t) &= -\frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} f(s, x_n(s), (Sx_n)(s)) ds, t \in J, \\ (F_3x)(t) &= x_0 + x_0^1 t + \frac{x_0^2}{2!} t^2 + \dots + \frac{x_0^{n-2}}{(n-2)!} t^{n-2} + \frac{x_T}{(n-1)!} t^{(n-1)}, t \in J. \end{aligned}$$

In view of hypothesis (H5) and lemma 2.6, the set $\overline{\text{conv}}K_1$ is compact. For any $t^* \in J$,

$$\begin{aligned} (F_1x_n)(t^*) &= \frac{1}{\Gamma(\alpha)} \int_0^{t^*} (t^*-s)^{\alpha-1} f(s, x_n(s), (Sx_n)(s)) ds \\ &= \frac{1}{\Gamma(\alpha)} \lim_{k \rightarrow \infty} \sum_{i=1}^k \frac{t^*}{k} \left(t^* - \frac{it^*}{k}\right)^{\alpha-1} f\left(\frac{it^*}{k}, x_n\left(\frac{it^*}{k}\right), (Sx_n)\left(\frac{it^*}{k}\right)\right) \\ &= \frac{t^*}{\Gamma(\alpha)} \zeta_{n1}, \end{aligned}$$

where

$$\zeta_{n1} = \lim_{k \rightarrow \infty} \sum_{i=1}^k \frac{1}{k} \left(t^* - \frac{it^*}{k}\right)^{\alpha-1} f\left(\frac{it^*}{k}, x_n\left(\frac{it^*}{k}\right), (Sx_n)\left(\frac{it^*}{k}\right)\right).$$

Now, we have $\{(F_1x_n)(t)\}$ is a function family of continuous mappings $F_1x_n : J \rightarrow X$, which is uniformly bounded and equicontinuous. As $\overline{\text{conv}}K_1$ is convex and compact, we know $\zeta_{n1} \in \overline{\text{conv}}K_1$. Hence, for any $t^* \in J = [0, T]$, the set $\{(F_1x_n)(t^*)\}$, is relatively compact. Therefore by lemma 2.7, every $\{(F_1x_n)(t)\}$ contains a uniformly convergent subsequence $\{(F_1x_{n_k})(t)\}, k = 1, 2, \dots$, on J . Thus, $\{F_1x : x \in B_{\eta^*}\}$ is relatively compact.

Set

$$(\overline{F}_2x_n)(t) = -\frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_0^t (t-s)^{\alpha-n} f(s, x_n(s), (Sx_n)(s)) ds, t \in J,$$

For any $t^* \in J$,

$$\begin{aligned}
(\overline{F}_2x_n)(t^*) &= -\frac{(t^*)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_0^{t^*} (t^*-s)^{\alpha-n} f(s, x_n(s), (Sx_n)(s)) ds \\
&= -\frac{(t^*)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \\
&\quad \times \lim_{k \rightarrow \infty} \sum_{i=1}^k \frac{t^*}{k} \left(t^* - \frac{it^*}{k}\right)^{\alpha-n} f\left(\frac{it^*}{k}, x_n\left(\frac{it^*}{k}\right), (Sx_n)\left(\frac{it^*}{k}\right)\right) \\
&= -\frac{(t^*)^n}{(n-1)!\Gamma(\alpha-n+1)} \zeta_{n2},
\end{aligned}$$

where

$$\zeta_{n2} = \lim_{k \rightarrow \infty} \sum_{i=1}^k \frac{1}{k} \left(t^* - \frac{it^*}{k}\right)^{\alpha-n} f\left(\frac{it^*}{k}, x_n\left(\frac{it^*}{k}\right), (Sx_n)\left(\frac{it^*}{k}\right)\right).$$

Now, we have $\{(\overline{F}_2x_n)(t)\}$ is a function family of continuous mappings $\overline{F}_2x_n : J \rightarrow X$, which is uniformly bounded and equicontinuous. As $\overline{\text{conv}}K_2$ is convex and compact, we know $\zeta_{n2} \in \overline{\text{conv}}K_2$. Hence, for any $t^* \in J = [0, T]$, the set $\{(\overline{F}_2x_n)(t^*)\}$, is relatively compact. Therefore by lemma 2.7, every $\{(\overline{F}_2x_n)(t)\}$ contains a uniformly convergent subsequence $\{(\overline{F}_2x_{n_k})(t)\}$, $k = 1, 2, \dots$, on J . Particularly, $\{(F_2x_n)(t)\}$ contains a uniformly convergent subsequence $\{(F_2x_{n_k})(t)\}$, $k = 1, 2, \dots$, on J . Thus, $\{F_2x : x \in B_{\eta^*}\}$ is relatively compact.

Obviously, the set $\{F_3x : x \in B_{\eta^*}\}$ is relatively compact. As a result, the set $\{Fx : x \in B_{\eta^*}\}$ is relatively compact.

As a consequence of steps 1-3, we can conclude that F is continuous and completely continuous.

Step 4. A priori bounds.

Now it remains to show that the set

$$E(F) = \{x \in C(J, X) : x = \eta Fx \text{ for some } \eta \in [0, 1]\},$$

is bounded

Let $x \in E(F)$, then $x = \eta Fx$ for some $\eta \in [0, 1]$. Thus, for each $t \in J$, we have

$$\begin{aligned}
x(t) = & \eta \left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), (Sx)(s)) ds \right. \\
& - \frac{t^{n-1}}{(n-1)! \Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} f(s, x(s), (Sx)(s)) ds \\
& \left. + x_0 + x_0^1 t + \frac{x_0^2}{2!} t^2 + \dots + \frac{x_0^{n-2}}{(n-2)!} t^{n-2} + \frac{x_T}{(n-1)!} t^{(n-1)} \right).
\end{aligned}$$

Using (H4), for each $t \in J$, we have

$$\begin{aligned}
\|x(t)\| & \leq \|(F(x))(t)\| \\
& \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} N_f (1 + \|x(s)\|^\lambda + N_k (1 + \|x_s\|_B^\lambda) T) ds \\
& \quad + \frac{t^{n-1}}{(n-1)! \Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} \\
& \quad \times N_f (1 + \|x(s)\|^\lambda + N_k (1 + \|x_s\|_B^\lambda) T) ds \\
& \quad + \|x_0\| + \|x_0^1\| t + \frac{\|x_0^2\|}{2!} t^2 + \dots + \frac{\|x_0^{n-2}\|}{(n-2)!} t^{n-2} + \frac{\|x_T\|}{(n-1)!} t^{n-1} \\
& \leq \frac{N_f}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \frac{N_f}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x(s)\|^\lambda ds \\
& \quad + \frac{N_f N_k T}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \frac{N_f N_k T}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_s\|_B^\lambda ds \\
& \quad + \frac{t^{n-1} N_f}{(n-1)! \Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} ds \\
& \quad + \frac{t^{n-1} N_f}{(n-1)! \Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} \|x(s)\|^\lambda ds \\
& \quad + \frac{t^{n-1} N_f N_k T}{(n-1)! \Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} ds \\
& \quad + \frac{t^{n-1} N_f N_k T}{(n-1)! \Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} \|x_s\|_B^\lambda ds \\
& \quad + \|x_0\| + \|x_0^1\| t + \frac{\|x_0^2\|}{2!} t^2 + \dots + \frac{\|x_0^{n-2}\|}{(n-2)!} t^{n-2} + \frac{\|x_T\|}{(n-1)!} t^{n-1} \\
& \leq \frac{N_f T^\alpha}{\Gamma(\alpha+1)} + \frac{N_f N_k T^{\alpha+1}}{\Gamma(\alpha+1)} + \frac{N_f T^\alpha}{(n-1)! \Gamma(\alpha-n+2)} + \frac{N_f N_k T^{\alpha+1}}{(n-1)! \Gamma(\alpha-n+2)} \\
& \quad + \|x_0\| + \|x_0^1\| T + \frac{\|x_0^2\|}{2!} T^2 + \dots + \frac{\|x_0^{n-2}\|}{(n-2)!} T^{n-2} + \frac{\|x_T\|}{(n-1)!} T^{n-1} \\
& \quad + \frac{N_f}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x(s)\|^\lambda ds + \frac{N_f N_k T}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_s\|_B^\lambda ds
\end{aligned}$$

$$\begin{aligned}
& + \frac{T^{n-1}N_f}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} \|x(s)\|^\lambda ds \\
& + \frac{T^n N_f N_k}{(n-1)!\Gamma(\alpha-n+1)} \int_0^T (T-s)^{\alpha-n} \|x_s\|_B^\lambda ds.
\end{aligned}$$

By lemma 3.2, there exists a $N > 0$ such that $\|x(t)\| \leq N, t \in J$.

Thus for every $t \in J$, we have $\|x\|_\infty \leq N$. This show that the set $E(F)$ is bounded.

As a consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point that is solution of fractional BVP (1). \square

5. EXAMPLES

In this section, we give tow examples to illustrate the usefulness of our main results.

Example 5.1.

$$(10) \quad \begin{cases} {}^c D^\alpha x(t) = \frac{e^{-\mu t}}{1+e^t} \left(\frac{\cos(t)|x(t)|}{1+|x(t)|} + \int_0^t \frac{e^{-\mu t}(s+|x(s)|)}{2(1+s+|x(s)|)} ds \right), t \in J_1, \alpha \in (3, 4), \\ x(0) = 0, x'(0) = 0, x''(0) = 0, x'''(1) = 0, \end{cases}$$

where $\mu > 0$ is constant.

Take $X_1 = [0, \infty)$, $J_1 = [0, 1]$ and so $T = 1$.

Set

$$f_1(t, x(t), (Sx)(t)) = \frac{e^{-\mu t}}{1+e^t} \left(\frac{\cos(t)|x(t)|}{1+|x(t)|} + (Sx)(t) \right), \quad k_1(t, s, x(s)) = \frac{e^{-\mu t}(s+|x(s)|)}{2(1+s+|x(s)|)}.$$

Let $x_1, x_2 \in C(J_1, X_1)$ and $t \in [0, 1]$, we have

$$\begin{aligned}
|k_1(t, s, x_1(s)) - k_1(t, s, x_2(s))| & \leq \frac{e^{-\mu t}}{2} \left| \frac{|x_1(s)| - |x_2(s)|}{(1+s+|x_1(s)|)(1+s+|x_2(s)|)} \right| \\
& \leq \frac{e^{-\mu t}}{2} |x_1(s) - x_2(s)|,
\end{aligned}$$

and

$$\begin{aligned}
& |f_1(t, x_1(t), (Sx_1)(t)) - f_1(t, x_2(t), (Sx_2)(t))| \\
& \leq \frac{e^{-\mu t}}{1+e^t} \left(|\cos(t)| \left| \frac{|x_1(t)|}{1+|x_1(t)|} - \frac{|x_2(t)|}{1+|x_2(t)|} \right| + |(Sx_1)(t) - (Sx_2)(t)| \right)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{e^{-\mu t}}{2} \left(\left| \frac{|x_1(t)| - |x_2(t)|}{(1 + |x_1(t)|)(1 + |x_2(t)|)} \right| + \left| (Sx_1)(t) - (Sx_2)(t) \right| \right) \\
&\leq \frac{e^{-\mu t}}{2} \left(|x_1(t) - x_2(t)| + \left| (Sx_1)(t) - (Sx_2)(t) \right| \right).
\end{aligned}$$

Also, for all $x \in C(J_1, X_1)$ and each $t \in J_1$, we have

$$\begin{aligned}
|f_1(t, x(t), (Sx)(t))| &\leq \frac{e^{-\mu t}}{1 + e^t} \left(\left| \frac{|x(t)|}{1 + |x(t)|} \right| + \frac{e^{-\mu t}}{2} \int_0^t \left| \frac{s + |x(s)|}{(1 + s + |x(s)|)} \right| ds \right) \\
&\leq \frac{e^{-\mu t}}{2} \left(1 + \frac{e^{-\mu t}}{2} \right).
\end{aligned}$$

For $t \in J_1, \beta \in (0, \alpha - 3)$, we have

$$\begin{aligned}
m_1(t) = \frac{e^{-\mu t}}{2} \in L^{\frac{1}{\beta}}(J_1, \mathbb{R}), m_2(t) = \frac{e^{-\mu t}}{2} \in L^{\frac{1}{\beta}}(J_1, \mathbb{R}), h(t) = \frac{e^{-\mu t}}{2} \left(1 + \frac{e^{-\mu t}}{2} \right) \in L^{\frac{1}{\beta}}(J_1, \mathbb{R}) \text{ and} \\
M = \left\| \frac{e^{-\mu t}}{2} + \frac{e^{-\mu t}}{2} \frac{e^{-\mu t}}{2} \right\|_{L^{\frac{1}{\beta}}(J_1, \mathbb{R})}.
\end{aligned}$$

Choosing some $\mu > 0$ large enough and suitable $\beta \in (0, \alpha - 3)$, one can arrive at the following inequality

$$\Omega_{\alpha, \beta} = \frac{M}{\Gamma(\alpha)} \frac{1}{\left(\frac{\alpha - \beta}{1 - \beta}\right)^{1 - \beta}} + \frac{M}{3! \Gamma(\alpha - 3)} \frac{1}{\left(\frac{\alpha - \beta - 3}{1 - \beta}\right)^{1 - \beta}} < 1.$$

All the assumptions in Theorem 4.1 are satisfied, and therefore, the fractional BVP 10 has a unique solution on J_1 .

Example 5.2.

$$(11) \quad \begin{cases} {}^c D^\alpha x(t) = \frac{t^\nu}{1 + e^t} \left(\frac{\cos(t)|x(t)|^\lambda}{1 + |x(t)|} + \int_0^t \frac{t^\nu |x(s)|^\lambda}{2(1 + |x(s)|)} ds \right), t \in J_1, \alpha \in (3, 4), \\ x(0) = 0, x'(0) = 0, x''(0) = 0, x'''(1) = 0, \\ \nu > -\alpha, \lambda \in [0, 1 - \frac{1}{p}), 1 < p < \frac{1}{4 - \alpha}. \end{cases}$$

Take $X_1 = [0, \infty)$, $J_1 = [0, 1]$ and so $T = 1$

Set

$$f_2(t, x(t), (Sx)(t)) = \frac{t^\nu}{1 + e^t} \left(\frac{\cos(t)|x(t)|^\lambda}{1 + |x(t)|} + (Sx)(t) \right), \quad k_2(t, s, x(s)) = \frac{t^\nu |x(s)|^\lambda}{2(1 + |x(s)|)},$$

$$|x| \leq M_1.$$

For all $x \in C(J_1, X_1)$ and each $t \in J_1 = [0, 1]$, we have

$$|k_2(t, s, x(s))| \leq \frac{1}{2} \left(1 + |x(s)|^\lambda \right),$$

and

$$\begin{aligned} |f_2(t, x(t), Sx(t))| &\leq \frac{t^\nu}{1 + e^t} \left(\left| \frac{|x(t)|^\lambda}{1 + |x(t)|} \right| + |(Sx)(t)| \right) \\ &\leq \frac{t^\nu}{2} (1 + |x(t)|^\lambda + |(Sx)(t)|) \\ &\leq \frac{1}{2} (1 + |x(t)|^\lambda + |(Sx)(t)|). \end{aligned}$$

Now, since $\nu > -\alpha$, we have

$$\begin{aligned} \int_0^t (t-s)^{\alpha-1} f_2(s, x(s), Sx(s)) ds &\leq \int_0^t (t-s)^{\alpha-1} s^\nu \left(|x(s)|^\lambda + \frac{s^\nu}{2} \int_0^s |x(\tau)|^\lambda d\tau \right) ds \\ &\leq M_1^\lambda \int_0^t (t-s)^{\alpha-1} \left(s^\nu + \frac{s^{2\nu}}{2} \right) ds \\ &\leq M_1^\lambda \Gamma(\alpha) I^\alpha \left(t^\nu + \frac{t^{2\nu}}{2} \right) \\ &\leq \frac{M_1^\lambda \Gamma(\alpha) \Gamma(1+\nu)}{\Gamma(1+\nu+\alpha)} + \frac{M_1^\lambda \Gamma(\alpha) \Gamma(1+2\nu)}{2\Gamma(1+2\nu+\alpha)}, \end{aligned}$$

and

$$\begin{aligned} \int_0^t (t-s)^{\alpha-4} f_2(s, x(s), Sx(s)) ds &\leq \int_0^t (t-s)^{\alpha-4} s^\nu \left(|x(s)|^\lambda + \frac{s^\nu}{2} \int_0^s |x(\tau)|^\lambda d\tau \right) ds \\ &\leq M_1^\lambda \int_0^t (t-s)^{\alpha-4} \left(s^\nu + \frac{s^{2\nu}}{2} \right) ds \\ &\leq M_1^\lambda \Gamma(\alpha-3) I^{\alpha-3} \left(t^\nu + \frac{t^{2\nu}}{2} \right) \\ &\leq \frac{M_1^\lambda \Gamma(\alpha-3) \Gamma(1+\nu)}{\Gamma(\nu+\alpha-2)} + \frac{M_1^\lambda \Gamma(\alpha-3) \Gamma(1+2\nu)}{2\Gamma(2\nu+\alpha-2)}. \end{aligned}$$

As a result, the sets

$$K_{11} = \left\{ (t-s)^{\alpha-1} f_2(s, x(s), Sx(s)) : x \in C(J_1, X_1), s \in [0, t] \right\},$$

$$K_{12} = \left\{ (t-s)^{\alpha-4} f_2(s, x(s), Sx(s)) : x \in C(J_1, X_1), s \in [0, t] \right\},$$

are bounded which implies that K_{11}, K_{12} are relatively compact. Thus, all the assumptions in Theorem 4.2 satisfied, and, hence, the fractional BVP 11 has at least one solution on J_1 .

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