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SOME GENERALIZED COMMON FIXED POINT THEOREMS UNDER RATIONAL CONTRACTIONS IN COMPLEX VALUED B-METRIC SPACES AND APPLICATIONS

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ABSTRACT. Recently, Rao et al. [9] introduced the notion of complex valued b-metric spaces, which was more general than the well known complex valued metric spaces that was introduced in 2011 by Azam et al.[1] In this paper, we prove some generalized common fixed point theorems satisfying contractive conditions involving rational expressions in complex valued b-metric spaces. Our results generalize and extend some of the known results in the literature. In the last section, we use our result to obtain the unique common solution of system of Urysohn integral equation.

1. INTRODUCTION AND PRELIMINARIES

Azam et al. [1], introduced the concept of complex valued metric spaces and established the existence of fixed point theorems under the contraction condition. Bhatt et al.[2] proved common fixed of mappings satisfying rational inequality in complex valued metric space. In 2012, Rouzkard and Imdad [8] extended and improved the results of Azam et al. [1]. Sintunavarat and Kumam [10] obtained common fixed point results by replacing constant of contractive condition to control functions.

In 2013, Rao et al. [9] introduced the notion of complex valued b-metric space which is a generalization of complex valued metric spaces. Since then, Aiman A. Mukheimer [6,7] obtained some common fixed point theorems in complex valued b-metric spaces. Subsequently, Dubey [4] and Dubey et al. [3,5] extended and generalize the results of Rao et al. [9] and A Mukheimer [6,7].

The aim of this paper is to obtain some common fixed point theorems for mappings satisfying a rational type contractive condition, in which the constant has been replaced by control functions; in the framework of complex valued b-metric spaces. The obtained results are generalizations of recent results proved by Aiman A. Mukheimer [6], Rouzkard and Imdad [8] and Sintunavarat and Kumam [10].

We recall some notations and definitions that will be needed in sequel.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2)$.

Thus $z_1 \precsim z_2$ if one of the following holds:

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(C1) $Re(z_1) = Re(z_2)\, and\, Im(z_1) < Im(z_2)$;**:**
(C2) $Re(z_1) < Re(z_2)\, and\, Im(z_1) = Im(z_2)$;**:**
(C3) $Re(z_1) < Re(z_2)\, and\, Im(z_1) < Im(z_2)$;**:**
(C4) $Re(z_1) = Re(z_2)\, and\, Im(z_1) = Im(z_2)$.**:**

In particular, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (C1),(C2) and(C3) is satisfied and we will write $z_1 \prec z_2$ if only (C3) is satisfied.

Remark 1.1. We obtained that the following statements hold:

(i)**:** $a, b \in \mathbb{R}$ and $a \leq b \Rightarrow az \precsim bz \forall z \in \mathbb{C}$.
(ii)**:** $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$.
(iii)**:** $z_1 \precsim z_2\, and\, z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

Definition 1.2[1] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:

(i)**:** $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(ii)**:** $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii)**:** $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Example 1.3[10] Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by $d(z_1, z_2) = e^{ik}|z_1 - z_2|$, where $k \in \mathbb{R}$. Then (X, d) is a complex valued metric space.

Definition 1.4[9] Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{C}$ is called complex valued b-metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(i)**:** $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
(ii)**:** $d(x, y) = d(y, x)$;
(iii)**:** $d(x, y) \precsim s[d(x, z) + d(z, y)]$.
The: pair (X, d) is called a complex valued b-metric space.
Example: 1.5[9] Let $X = [0, 1]$. Define the mapping $d : X \times X \to \mathbb{C}$ by

$d(x, y) = |x - y|^2 + i|x - y|^2$, for all $x, y \in X$.

Then (X, d) is a complex valued b-metric space with $s = 2$.

Definition 1.6[9] Let (X, d) be a complex valued b-metric space.

(i) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A$.

(ii) A point $x \in X$ is called a limit point of a set A whenever for every $0 \prec r \in \mathbb{C}, B(x, r) \cap (A - \{x\}) \neq \phi$.

(iii) A subset $A \subseteq X$ is called open set whenever each element of A is an interior point of a set A.

(iv) A subset $A \subseteq X$ is called closed set whenever each limit point of A belongs to A.

(v) A sub-basis for a Hausdorff topology τ on X is a family $F = \{B(x,r) : x \in X\, and\, 0 \prec r\}$.

Definition 1.7[**9**] Let (X,d) be a complex valued b-metric space, $\{x_n\}$be a sequence in X and $x \in X$.

(i) If for every $c \in \mathbb{C}$, with $0 \prec c$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$converges to x and x is the limit point of $\{x_n\}$. We denote this by $lim_{n\to\infty} x_n = x$ or $\{x_n\} \to x\, as\, n \to \infty$.

(ii) If for every $c \in \mathbb{C}$, with $0 \prec c$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$is said to be Cauchy sequence.

(iii) If every Cauchy sequence in X is convergent, then (X,d) is said to be a complete complex valued b-metric space.

Lemma 1.8[**9**] Let (X,d) be a complex valued b-metric space and let $\{x_n\}$be a sequence in X. Then $\{x_n\}$converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma 1.9[**9**] Let (X,d) be a complex valued b-metric space and let $\{x_n\}$be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.

2. MAIN RESULTS

Theorem 2.1. Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S, T : X \to X$. If there exists a mapping $\lambda, \mu, \gamma : X \to [0,1)$ such that for all $x, y \in X$:

(i) $\lambda(Sx) \leq \lambda(x), \mu(Sx) \leq \mu(x)\, and\, \gamma(Sx) \leq \gamma(x)$;

(ii) $\lambda(Tx) \leq \lambda(x), \mu(Tx) \leq \mu(x)\, and\, \gamma(Tx) \leq \gamma(x)$;

(iii) $s\lambda(x) + \mu(x) + \gamma(x) < 1$;

(iv) $d(Sx, Ty) \precsim \lambda(x)d(x,y) + \frac{\mu(x)d(x,Sx)d(y,Ty)+\gamma(x)d(y,Sx)d(x,Ty)}{1+d(x,y)}. --- (2.1)$

Then S and T have a unique common fixed point.

Proof. For any arbitrary point $x_0 \in X$. Since $S(X) \subseteq X$ and $T(X) \subseteq X$, we can define sequence $\{x_n\}$ in X such that

$x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1},\, for\, n \geq 0. --- (2.2)$

Now, we show that the sequence $\{x_n\}$ is Cauchy. Let $x = x_{2n}$ and $y = x_{2n+1}$ in (2.1), we get

$$
\begin{aligned}
& d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1}) \\
& \precsim \lambda(x_{2n})d(x_{2n}, x_{2n+1}) + \tfrac{\mu(x_{2n})d(x_{2n},Sx_{2n})d(x_{2n+1},Tx_{2n+1})+\gamma(x_{2n})d(x_{2n+1},Sx_{2n})d(x_{2n},Tx_{2n+1})}{1+d(x_{2n},x_{2n+1})} \\
& = \lambda(x_{2n})d(x_{2n}, x_{2n+1}) + \tfrac{\mu(x_{2n})d(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+2})+\gamma(x_{2n})d(x_{2n+1},x_{2n+1})d(x_{2n},x_{2n+2})}{1+d(x_{2n},x_{2n+1})} \\
& \precsim \lambda(x_{2n})d(x_{2n}, x_{2n+1}) + \mu(x_{2n})d(x_{2n+1}, x_{2n+2}) \left(\tfrac{d(x_{2n},x_{2n+1})}{1+d(x_{2n},x_{2n+1})}\right) \\
& \precsim \lambda(x_{2n})d(x_{2n}, x_{2n+1}) + \mu(x_{2n})d(x_{2n+1}, x_{2n+2}) \\
& = \lambda(Tx_{2n-1})d(x_{2n}, x_{2n+1}) + \mu(Tx_{2n-1})d(x_{2n+1}, x_{2n+2}) \\
& \precsim \lambda(x_{2n-1})d(x_{2n}, x_{2n+1}) + \mu(x_{2n-1})d(x_{2n+1}, x_{2n+2}) \\
& = \lambda(Sx_{2n-2})d(x_{2n}, x_{2n+1}) + \mu(Sx_{2n-2})d(x_{2n+1}, x_{2n+2}) \\
& \precsim \lambda(x_{2n-2})d(x_{2n}, x_{2n+1}) + \mu(x_{2n-2})d(x_{2n+1}, x_{2n+2})
\end{aligned}
$$

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$\precsim \lambda(x_0)d(x_{2n}, x_{2n+1}) + \mu(x_0)d(x_{2n+1}, x_{2n+2})$
which implies that
$d(x_{2n+1}, x_{2n+2}) \precsim \left(\frac{\lambda(x_0)}{1-\mu(x_0)}\right) d(x_{2n}, x_{2n+1}). ---(2.3)$
Similarly, we obtain
$d(x_{2n+2}, x_{2n+3}) \precsim \left(\frac{\lambda(x_0)}{1-\mu(x_0)}\right) d(x_{2n+1}, x_{2n+2}). ---(2.4)$
Putting $h := \frac{\lambda(x_0)}{1-\mu(x_0)}$ for all $n \geq 0$, we have
$d(x_{2n+1}, x_{2n+2}) \precsim hd(x_{2n}, x_{2n+1}) \precsim h^2 d(x_{2n-1}, x_{2n}) \precsim ----$
$----- \precsim h^{2n+1}d(x_0, x_1)$
$i.e$ $d(x_{n+1}, x_{n+2}) \precsim hd(x_n, x_{n+1}) \precsim h^2 d(x_{n-1}, x_n) \precsim --- \precsim h^{n+1}d(x_0, x_1)$
or $d(x_n, x_{n+1}) \precsim h^n d(x_0, x_1). ---(2.5)$
Thus for any $m > n, m, n \in \mathbb{N}$ and since
$sh = \frac{s\lambda(x_0)}{1-\mu(x_0)} < 1$, we get
$d(x_n, x_m) \precsim sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m)$
$\precsim sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_m)$
$-----------------------------$
$-----------------------------$
$\precsim sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) + ---$
$--- + s^{m-n-1}d(x_{m-2}, x_{m-1}) + s^{m-n}d(x_{m-1}, x_m).$
By using (2.5), we get
$d(x_n, x_m) \precsim sh^n d(x_0, x_1) + s^2 h^{n+1} d(x_0, x_1) + --- + s^{m-n}h^{m-1}d(x_0, x_1)$
$$= \sum_{i=1}^{m-n} s^i h^{i+n-1} d(x_0, x_1).$$
Therefore,
$$d(x_n, x_m) \precsim \sum_{i=1}^{m-n} s^{i+n-1} h^{i+n-1} d(x_0, x_1)$$
$$= \sum_{t=n}^{m-1} s^t h^t d(x_0, x_1)$$
$$\precsim \sum_{t=n}^{\infty} (sh)^t d(x_0, x_1)$$
$= \frac{(sh)^n}{1-sh} d(x_0, x_1). ---(2.6)$
Therefore,
$|d(x_n, x_m)| \leq \frac{(sh)^n}{1-sh} |d(x_0, x_1)| \to 0$ as $m, n \to \infty$.

Thus $\{x_n\}$ is a Cauchy sequence in X. By completeness of X, there exists a point $u \in X$ such that $x_n \to u$ as $n \to \infty$. Next, we claim that $Su = u$.

Assume not, then there exists $z \in X$ such that
$|d(u, Su)| = |z| > 0. - - - (2.7)$
So by using the notion of a complex valued b-metric, we have
$z = d(u, Su)$
$\precsim sd(u, x_{2n+2}) + sd(x_{2n+2}, Su)$
$= sd(u, x_{2n+2}) + sd(Su, Tx_{2n+1})$
$\precsim sd(u, x_{2n+2}) + s\lambda(u)d(u, x_{2n+1})$
$+ \frac{s\mu(u)d(u,Su)d(x_{2n+1},Tx_{2n+1})+s\gamma(u)d(x_{2n+1},Su)d(u,Tx_{2n+1})}{1+d(u,x_{2n+1})}$
$= sd(u, x_{2n+2}) + s\lambda(u)d(u, x_{2n+1})$
$+ \frac{s\mu(u)d(u,Su)d(x_{2n+1},x_{2n+2})+s\gamma(u)d(x_{2n+1},Su)d(u,x_{2n+2})}{1+d(u,x_{2n+1})}$
which implies that
$|z| = |d(u, Su)| \leq s|d(u, x_{2n+2})| + s\lambda(u)|d(u, x_{2n+1})|$
$+ \frac{s\mu(u)|z||d(x_{2n+1}.x_{2n+2})|+s\gamma(u)|d(x_{2n+1},Su)||d(u,x_{2n+2})|}{|1+d(u,x_{2n+1})|}. - - - (2.8)$
Taking the limit of (2.8) as $n \to \infty$, we get that $|z| = |d(u, Su)| \leq 0$, a contradiction with (2.7). So $|z| = 0$.

Hence $Su = u$. It follows similarly $Tu = u$. Therefore, u is common fixed point of S and T.

Finally, we show that u is a unique common fixed point of S and T. Assume that there exists another common fixed point $u^\star$ that is $u^\star = Su^\star = Tu^\star$. Then
$d(u, u^\star) = d(Su, Tu^\star)$
$\precsim \lambda(u)d(u, u^\star) + \frac{\mu(u)d(u,Su)d(u^\star,Tu^\star)+\gamma(u)d(u^\star Su)d(u,Tu^\star)}{1+d(u,u^\star)}$
$= \lambda(u)d(u, u^\star) + \frac{\gamma(u)d(u,u^\star)d(u,u^\star)}{1+d(u,u^\star)}$.
So that
$|d(u, u^\star)| \leq \lambda(u)|d(u, u^\star)| + \frac{\gamma(u)|d(u^\star,u)||d(u,u^\star)|}{|1+d(u,u^\star)|}$.
Since $|1 + d(u, u^\star)| > |d(u, u^\star)|$, therefore
$|d(u, u^\star)| \leq (\lambda + \gamma)(u)|d(u, u^\star)|$.
Since $\lambda(u)\, and\, \gamma(u) \in [0, 1)$, we have $|d(u, u^\star)| = 0$.
Therefore, we have $u = u^\star$ and thus u is a unique common fixed point of S and T.

Corollary 2.2. Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S, T : X \to X$ are mappings satisfying:
$d(Sx, Ty) \precsim \lambda d(x, y) + \frac{\mu d(x,Sx)d(y,Ty)+\gamma d(y,Sx)d(x,Ty)}{1+d(x,y)} - - - (2.9)$
for all $x, y \in X$, where λ, μ, γ are nonnegative reals with $s\lambda + \mu + \gamma < 1$. Then S and T have a unique common fixed point.

Proof. We can prove this result by applying Theorem 2.1 by setting $\lambda(x) = \lambda, \mu(x) = \mu$ and $\gamma(x) = \gamma$.

Corollary 2.3. Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$. If there exists a mapping $\lambda, \mu, \gamma : X \to [0, 1)$ such that for all $x, y \in X$:

(i) $\lambda(Tx) \leq \lambda(x), \mu(Tx) \leq \mu(x)\, and\, \gamma(Tx) \leq \gamma(x)$;

(ii) $s\lambda(x) + \mu(x) + \gamma(x) < 1$;

(iii) $d(Tx, Ty) \precsim \lambda(x)d(x, y)+\frac{\mu(x)d(x,Tx)d(y,Ty)+\gamma(x)d(y,Tx)d(x,Ty)}{1+d(x,y)}$. $---$ (2.10)

Then T has a unique fixed point.

Proof. We can prove this result by applying Theorem 2.1 with $S = T$.

Corollary 2.4. Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mapping satisfying:

$d(Tx, Ty) \precsim \lambda d(x, y) + \frac{\mu d(x,Tx)d(y,Ty)+\gamma d(y,Tx)d(x,Ty)}{1+d(x,y)}$ $---$ (2.11)

for all $x, y \in X$, where λ, μ, γ are nonnegative reals with $s\lambda + \mu + \gamma < 1$. Then T has a unique fixed point.

Proof. We can prove this result by applying Corollary 2.3 with $\lambda(x) = \lambda, \mu(x) = \mu\, and\, \gamma(x) = \gamma$.

Corollary 2.5. Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$. If there exists a mapping $\lambda, \mu, \gamma : X \to [0, 1)$ such that for all $x, y \in X$ and for some $n \in \mathbb{N}$:

(i) $\lambda(T^n x) \leq \lambda(x), \mu(T^n x) \leq \mu(x)\, and\, \gamma(T^n x) \leq \gamma(x)$;

(ii) $s\lambda(x) + \mu(x) + \gamma(x) < 1$;

(iii) $d(T^n x, T^n y) \precsim \lambda(x)d(x, y) + \frac{\mu(x)d(x,T^n x)d(y,T^n y)+\gamma(x)d(y,T^n x)d(x,T^n y)}{1+d(x,y)}$.
$---$(2.12)

Then T has a unique fixed point.

Proof: From Corollary 2.3, we get T^n has a unique fixed point u. It follows from

$T^n(Tu) = T(T^n u) = Tu$

that is Tu is a fixed point of T^n. Therefore $Tu = u$ by the uniqueness of a fixed point of T^n and then u is also a fixed point of T. Since the fixed point of T is also fixed point of T^n, the fixed point of T is unique. The uniqueness follows from

$d(Tu, u) = d(TT^n u, T^n u) = d(T^n Tu, T^n u) = d(T^n u, T^n Tu)$

$\precsim \lambda(u)d(u, Tu) + \frac{\mu(u)d(u,T^n u)d(Tu,T^n Tu)+\gamma(u)d(Tu,T^n u)d(u,T^n Tu)}{1+d(u,Tu)}$

$= \lambda(u)d(u, Tu) + \frac{\mu(u)d(u,T^n u)d(Tu,TT^n u)+\gamma(u)d(Tu,T^n u)d(u,TT^n u)}{1+d(u,Tu)}$

$\precsim \lambda(u)d(u, Tu) + \frac{\gamma(u)d(u,Tu)d(u,Tu)}{1+d(u,Tu)}$.

So that

$|d(Tu, u)| \leq \lambda(u)|d(u, Tu)| + \frac{\gamma(u)|d(u,Tu)||d(u,Tu)|}{|1+d(u,Tu)|}$.

Since $|1 + d(u, Tu)| > |d(u, Tu)|$.

Therefore,

$|d(Tu, u)| \leq (\lambda + \gamma)(u)|d(Tu, u)|$,

a contradiction. So $Tu = u$. Hence $Tu = T^n u = u$. Therefore the fixed point of T is unique. This completes the proof.

Corollary 2.6. Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mapping satisfying (for some fixed $n \in \mathbb{N}$):

$d(T^n x, T^n y) \precsim \lambda d(x,y) + \frac{\mu d(x,T^n x)d(y,T^n y)+\gamma d(y,T^n x)d(x,T^n y)}{1+d(x,y)} - - - (2.13)$

for all $x, y \in X$, where λ, μ, γ are nonnegative reals with $s\lambda + \mu + \gamma < 1$. Then T has a unique fixed point in X.

Proof. We can prove this result by applying Corollary 2.5 with $\lambda(x) = \lambda, \mu(x) = \mu \, and \, \gamma(x) = \gamma$.

Theorem 2.7.[10] Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S, T : X \to X$. If there exists a mapping $\lambda, \mu : X \to [0, 1)$ such that for all $x, y \in X$:

(i) $\lambda(Sx) \leq \lambda(x)$ and $\mu(Sx) \leq \mu(x)$;

(ii) $\lambda(Tx) \leq \lambda(x) \, and \, \mu(Tx) \leq \mu(x)$;

(iii) $s\lambda(x) + \mu(x) < 1$;

(iv) $d(Sx, Ty) \precsim \lambda(x)d(x,y) + \frac{\mu(x)d(x,Sx)d(y,Ty)}{1+d(x,y)}. - - - (2.14)$

Then S and T have a unique common fixed point.

Proof. We can prove this result by applying Theorem 2.1 with $\gamma(x) = 0$.

Corollary 2.8. Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S, T : X \to X$ are mappings satisfying:

$d(Sx, Ty) \precsim \lambda d(x,y) + \frac{\mu d(x,Sx)d(y,Ty)}{1+d(x,y)} - - - (2.15)$

for all $x, y \in X$, where λ, μ are nonnegative reals with $s\lambda + \mu < 1$. Then S and T have a unique common fixed point.

Proof. We can prove this result by applying Theorem 2.7 by setting $\lambda(x) = \lambda \, and \, \mu(x) = \mu$.

Corollary 2.9. Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$. If there exists a mapping $\lambda, \mu : X \to [0, 1)$ such that for all $x, y \in X$:

(i) $\lambda(Tx) \leq \lambda(x)$ and $\mu(Tx) \leq \mu(x)$;

(ii) $s\lambda(x) + \mu(x) < 1$;

(iii) $d(Tx, Ty) \precsim \lambda(x)d(x,y) + \frac{\mu(x)d(x,Tx)d(y,Ty)}{1+d(x,y)}. - - - (2.16)$

Then T has a unique fixed point.

Proof. We can prove this result by applying Theorem 2.7 with $S = T$.

Corollary 2.10. Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mapping satisfying:

$d(Tx, Ty) \precsim \lambda d(x,y) + \frac{\mu d(x,Tx)d(y,Ty)}{1+d(x,y)} - - - (2.17)$

for all $x, y \in X$, where λ, μ are nonnegative reals with $s\lambda + \mu < 1$. Then T has a unique fixed point.

Proof. We can prove this result by applying Corollary 2.9 with $\lambda(x) = \lambda$ and $\mu(x) = \mu$.

Corollary 2.11. Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$. If there exists a mapping $\lambda, \mu : X \to [0, 1)$ such that for all $x, y \in X$ and for some $n \in \mathbb{N}$:

(i) $\lambda(T^n x) \leq \lambda(x)$ and $\mu(T^n x) \leq \mu(x)$;

(ii) $s\lambda(x) + \mu(x) < 1$;

(iii) $d(T^n x, T^n y) \precsim \lambda(x)d(x,y) + \frac{\mu(x)d(x,T^n x)d(y,T^n y)}{1+d(x,y)}$. $---$ (2.18)

Then T has a unique fixed point.

Proof. The proof of this Corallary is similar as Corallary 2.5.

Corollary 2.12. Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mapping satisfying (for some fixed $n \in \mathbb{N}$) :

$d(T^n x, T^n y) \precsim \lambda d(x,y) + \frac{\mu d(x,T^n x)d(y,T^n y)}{1+d(x,y)}$ $---$ (2.19)

for all $x, y \in X$, where λ, μ are nonnegative reals with $s\lambda + \mu < 1$. Then T has a unique fixed point in X.

Proof. We can prove this result by applying Corollary 2.11 with $\lambda(x) = \lambda$ and $\mu(x) = \mu$.

3. APPLICATION

In this section, we use Theorem 2.7 to obtain an existence of common solution of the system of Urysohn integral equations.

Theorem 3.1.[10] Let $X = C([a,b], \mathbb{R}^n)$, where $[a,b] \subseteq \mathbb{R}^+$ and $d : X \times X \to \mathbb{C}$ be defined by

$d(x,y) = \max_{t \in [a,b]} ||x(t) - y(t)||_\infty \sqrt{1+a^2} e^{i tan^{-1} a}$.

Consider the Urysohn integral equations

$x(t) = \int_a^b K_1(t,p,x(p))dp + g(t),$ $---$(3.1)

$x(t) = \int_a^b K_2(t,p,x(p))dp + h(t),$ $---$(3.2)

where $t \in [a,b] \subseteq \mathbb{R}\, and\, x, g, h \in X$.

Suppose that $K_1, K_2 : [a,b] \times [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ are such that $F_x, G_x \in X$ for all $x \in X$, where

$F_x(t) = \int_a^b K_1(t,p,x(p))dp$

and

$G_x(t) = \int_a^b K_2(t,p,x(p))dp$

for all $t \in [a,b]$.

If there exists two mappings $\lambda, \mu : X \to [0,1)$ such that for all $x, y \in X$ the following holds:

(i) $\lambda(F_x + g) \leq \lambda(x)$ and $\mu(F_x + g) \leq \mu(x)$;

(ii) $\lambda(G_x + h) \leq \lambda(x)$ and $\mu(G_x + h) \leq \mu(x)$;

(iii) $s\lambda(x) + \mu(x) < 1$, where $s \geq 1$;

(iv) $||F_x(t) - G_y(t) + g(t) - h(t)||_\infty \sqrt{1+a^2} e^{i tan^{-1} a} \precsim \lambda(x)A(x,y)(t) + \mu(x)B(x,y)(t)$,

where $A(x,y)(t) = ||x(t) - y(t)||_\infty \sqrt{1+a^2} e^{i tan^{-1} a}$,

$B(x,y)(t) = \frac{||F_x(t)+g(t)-x(t)||_\infty ||G_y(t)+h(t)-y(t)||_\infty}{1+d(x,y)} \sqrt{1+a^2} e^{i tan^{-1} a}$,

then the system of integral equations (3.1) and (3.2) have a unique common solution.

Proof. Define two mappings $S, T : X \to X$ by $Sx = F_x + g$ and $Tx = G_x + h$. Then

$d(Sx,Ty) = \max_{t \in [a,b]} ||F_x(t) - G_y(t) + g(t) - h(t)||_\infty \sqrt{1+a^2} e^{i tan^{-1} a}$

$d(x,Sx) = \max_{t \in [a,b]} ||F_x(t) + g(t) - x(t)||_\infty \sqrt{1+a^2} e^{i tan^{-1} a}$

and

$d(y,Ty) = \max_{t\in[a,b]} ||Gy(t)+h(t)-y(t)||_{\infty}\sqrt{1+a^2}e^{itan^{-1}a}$.

We can show easily that for all $x, y \in X$,

$d(Sx,Ty) \precsim \lambda(x)d(x,y) + \frac{\mu(x)d(x,Sx)d(y,Ty)}{1+d(x,y)}$.

Now, we can apply Theorey 2.7, we get the Urysohn integral equations (3.1) and (3.2) have a unique common solution.

4. CONCLUSION

In this article, we prove some common fixed point theorems in complex valued b-metric spaces.These results generalizes and improves the recent results of Aiman A. Mukheimer [6], Rouzkard and Imdad [8] and Sintunavarat and Kumam [10], in the sense that in our results in contractive conditions, replacing the constants with functions, which extends the further scope of our results. For the usability of our results we give an application in the last section of this paper.

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