RATE OF CONVERGENCE BY A NEW CLASS OF STANCU GENERALIZED INTEGRAL OPERATORS

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Abstract. In this paper we will study about a new family of Stancu generalized operators defined on the class of all Lebesgue measurable functions f on $[0, \infty)$. Here we obtain some direct results using Taylor's expansion and study rate of convergence of integral operators in simultaneous approximation. Here with the use of differential operator we will also estimate moments for our operator.

1. INTRODUCTION

Motivated by the work of several authors [1] [2] [3] [4] [5] [6] [7] [8] on Szasz Mirakyan operators and their various modifications, we define a new class of Stancu type generalized operators, given by,

$$\left(B_{(n,r,\alpha,\beta)}f\right)(x) = \frac{n^r(n-r-1)!}{(n-2)!}\sum_{i=0}^{\infty} k_{(n,i)}(x) \int_0^{\infty} S_{(n-r,i+r)}(t)f\left(\frac{nt+\alpha}{n+\beta}\right)dt$$

$$\left(B_{(n,\alpha,\beta)}f\right)(x) = (n-1)\sum_{i=0}^{\infty} k_{(n,i)}(x) \int_0^{\infty} S_{(n,i)}(t)f\left(\frac{nt+\alpha}{n+\beta}\right)dt \qquad \text{for } r = 0 \qquad (1.1)$$

Where α and β are two non-negative parametrs satisfying the condition $0 \leq \alpha \leq \beta$ for any non- negative integer n.

Here, $k_{(n,i)}(x) = \frac{(nx)^i}{e^{nx}i!}$

$$S_{(n,i)}(t) = \binom{n+i-1}{i}t^i(1+t)^{-(n+i)} \qquad (1.2)$$

For $\alpha = 0, \ \beta = 0$, (1.1) becomes well known Szasz Mirakyan Baskakov opeators.

$$\left(B_{(n,0,0)}f\right)(x) = \left(B_{(n)}f\right)(x) = (n-1)\sum_{i=0}^{\infty} k_{(n,i)}(x) \int_0^{\infty} S_{(n,i)}(t)f(t)dt \qquad (1.3)$$

Let L be the class of all lebesgue integrable functions f on $[0,\infty)$ satisfying,

$\int_0^{\infty} \frac{|f(t)|}{(1+t)^n}dt < \infty$, for $n \in N$

Also, we can see that,

(i) $\sum_{i=0}^{\infty} S_{(n,i)}(t) = 1$

(ii) $\int_0^{\infty} S_{(n,i)}(t) = \frac{1}{(n-1)}$

(iii) $\sum_{i=0}^{\infty} k_{(n,i)}(x) = 1$

(iv) $\int_0^{\infty} k_{(n,i)}(x) = \frac{1}{n}$

The aim of this paper is to study approximation properties of stancu generalized operators. Here, we will estimate moments and study some direct results.

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2. ESTIMATION OF MOMENTS

Lemma 2.1. [9] For $\in N \cup \{0\}$, we have,

$$\mu_{(n,\alpha,\beta,m)}(x) = \sum_{i=0}^{\infty} k_{(n,i)}(x)\left((i/n) - x\right)^m \tag{2.1}$$

For this, recurrence relation is given by,

$$n\mu_{(n,\alpha,\beta,m+1)}(x) = x\mu'_{(n,\alpha,\beta,m)}(x) + xm\mu_{(n,\alpha,\beta,m-1)}(x) \tag{2.2}$$

Here from **[10] [11]**,

(i) $\mu_{(n,\alpha,\beta,m)}(x)$ is a polynomial in x.

(ii) $\mu_{(n,\alpha,\beta,m+1)}(x) = O\left(n^{-\left[\frac{(m+1)}{2}\right]}\right)$

Here, $[\alpha]$ denotes integral part of α.

Lemma 2.2. [12] Let the mth order moment be defined by,

$$T_{(r,n,m,\alpha,\beta)}(x) = B_{(n,r,\alpha,\beta)}\left(\left(\frac{nt+\alpha}{n+\beta} - x\right)^m, x\right)$$

$$= \frac{n^r(n-r-1)!}{(n-2)!}\sum_{i=0}^{\infty} k_{(n,i)}(x)\int_0^{\infty} S_{(n-r,i+r)}(t)\left(\frac{nt+\alpha}{n+\beta} - x\right)^m dt$$

Then,

$$T_{(n,0,\alpha,\beta)}(x) = 1$$

$$T_{(n,1,\alpha,\beta)}(x) = \frac{(2n-n\beta+2\beta)x+(1+\alpha)n-2\alpha}{(n-2)(n+\beta)} \quad , \text{for } n > 2, \tag{2.3}$$

$$T_{(n,2,\alpha,\beta)}(x) = \left(\frac{n^3+(\beta^2-4\beta+6)n^2+(12\beta-5\beta^2)n+6\beta^2}{(-2+n)(-3+n)(n+\beta)^2}\right)x^2$$

$$+ \left(\frac{2n^3+(6+4\alpha-2\beta-2\alpha\beta)n^2+(6\beta+10\alpha\beta-12\alpha)n-12\alpha\beta}{(-2+n)(-3+n)(n+\beta)^2}\right)x$$

$$+ \frac{(2+\alpha^2+2\alpha)n^2-(6\alpha+5\alpha^2)n+6\alpha^2}{(-2+n)(-3+n)(n+\beta)^2} \tag{2.4}$$

Lemma 2.3.[2] [3] [13] Let f be p times differentiable on $[0,\infty)$ such that

$f^{(p-1)}\left(\frac{nt+\alpha}{n+\beta}\right) = O\left(\left(\frac{nt+\alpha}{n+\beta}\right)^\gamma\right)$ for some $\gamma > 0$ as $t \to \infty$

Then for $p = 1,2,3, \ldots\ldots\ldots\ldots$, we have,

$$D^p\left(B_{(n,0,\alpha,\beta)}f\right)(x) = \frac{p(n)^{2p}(n-p-2)!}{(n-2)!\,(n+\beta)^p}\sum_{i=0}^{\infty} k_{(n,i)}(x)\int_0^{\infty} S_{(n,i)}(t)\,f^{(p)}\left(\frac{nt+\alpha}{n+\beta}\right)dt$$

Proof. We have the following equalities,

$B_{(n)}(1)(x) = 1$

And $B_{(n)}(t)(x) = \frac{1+nx}{(n-2)}$

Also, $B_{(n,0,\alpha,\beta)}(1)(x) = B_{(n)}(1)(x) = 1$

$B_{(n,0,\alpha,\beta)}(t)(x) = \frac{n}{(n+\beta)}B_{(n)}(t)(x) + \frac{\alpha}{(n+\beta)}B_{(n)}(1)(x)$

$$= \frac{n}{(n+\beta)} \left[\frac{1+nx}{(n-2)}\right] + \frac{\alpha}{(n+\beta)} [1]$$

$$= \frac{(1+nx+\alpha)n-2\alpha}{(n-2)(n+\beta)}$$

$$D^1 B_{(n,0,\alpha,\beta)}(t)(x) = \frac{n^2}{(n-2)(n+\beta)}$$

$$B_{(n)}(t^2)(x) = \frac{(nx^2+4x)n+2}{(n-2)(n-3)}$$

$$B_{(n,0,\alpha,\beta)}(t^2)(x) = \frac{n^2}{(n+\beta)^2} \left[\frac{(nx^2+4x)n+2}{(n-2)(n-3)}\right] + \frac{2n\alpha}{(n+\beta)^2} \left[\frac{1+nx}{(n-2)}\right] + \frac{\alpha^2}{(n+\beta)^2} [1]$$

$$= \frac{x^2n^4 + (4x+2\alpha x)n^3 + (-6\alpha x + \alpha^2 + 2\alpha + 2)n^2 + (-5\alpha^2 - 6\alpha)n + 6\alpha^2}{(n-2)(n-3)(n+\beta)^2}$$

$$D^2 B_{(n,0,\alpha,\beta)}(t^2)(x) = \frac{2n^4}{(n-2)(n-3)(n+\beta)^2}$$

On similar lines, we will get the desired result.

Lemma 2.4 [14]. There exist polynomials $\Psi_{(p,q,r)}(x)$ independent of n and i such that,

$$x^r D^r \left((nx)^i / e^{nx}\right) = \sum_{\substack{2p+q \leq r \\ p,q \geq 0}} n^p (i - nx)^q \Psi_{(p,q,r)}(x) \frac{(nx)^i}{e^{nx}}$$

Lemma 2.5. For every $x \in [0, \infty)$, we have,

$$\lim_{n \to \infty} n B_{(n,0,\alpha,\beta)}(t - x, x) = 1 + 2x + \alpha - \beta x$$

$$\lim_{n \to \infty} n B_{(n,0,\alpha,\beta)}((t - x)^2, x) = x^2 + 2x$$

3. DIRECT RESULTS

Theorem 3.1. [4] [15] [16] [17] Let f be any function bounded on every finite sub-intervalof $[0,\infty)$ and $f^{(1)}(x), f^{(2)}(x)$ exists at a fixed point $x \in [0, \infty)$. Let $f\left(\frac{nt+\alpha}{n+\beta}\right) = O\left(\frac{nt+\alpha}{n+\beta}\right)^\gamma$ as $t \to \infty$ for some $\gamma > 0$, then,

$$\lim_{n \to \infty} n\left[B_{(n,0,\alpha,\beta)}f(x) - f(x)\right] = [1 + 2x + \alpha - \beta x]f^{(1)}(x) + \left(\frac{x^2+2x}{2}\right)f^{(2)}(x)$$

for $f(x) = x^2$.

Proof. We can write this identity by using Taylor's expansion and [18] [19] [20],

$$f(t) - f(x) = (t - x)f^{(1)}(x) + \frac{1}{2!}(t - x)^2 f^{(2)}(x) + R(t, x)(t - x)^2$$

Here, $R(t, x)$ is Peano form of remainder and,

$$\lim_{t \to x} R(t, x) = 0 \text{ Now,}$$

$$n\left[(B_{(n,0,\alpha,\beta)}f)(x) - f(x)\right] = nf^{(1)}(x)B_{(n,0,\alpha,\beta)}(t - x, x) + \frac{n}{2!}f^{(2)}(x)B_{(n,0,\alpha,\beta)}((t - x)^2, x)$$

$$+ nB_{(n,0,\alpha,\beta)}(R(t, x)(t - x)^2, x)$$

Using Cauchy-schwarz inequality, we have,

$$B_{(n,0,\alpha,\beta)}(R(t,x)(t-x)^2,x) \le \left(B_{(n,0,\alpha,\beta)}(R(t,x)^2,x)\right)^{1/2}\left(B_{(n,0,\alpha,\beta)}((t-x)^4,x)\right)^{1/2} \quad (3.1)$$

Here, $R^2(x,x) = 0$,

We can see that,

$$\lim_{n \to \infty} nB_{(n,0,\alpha,\beta)}(R(t,x)^2,x) = R^2(x,x) = 0 \qquad\qquad (3.2)$$

Uniformly with respect to $x \in [0,A]$, where, $A > 0$.

From (3.1) , (3.2) and lemma 2.5 , we have,

$$\lim_{n \to \infty} nB_{(n,0,\alpha,\beta)}(R(t,x)(t-x)^2,x) = 0$$

Hence,

$$\lim_{n \to \infty} n[(B_{(n,0,\alpha,\beta)}f)(x) - f(x)] = \lim_{n \to \infty}\left[nf^{(1)}(x)B_{(n,0,\alpha,\beta)}(t-x,x) + \right.$$

$$\frac{n}{2!}f^{(2)}(x)B_{(n,0,\alpha,\beta)}((t-x)^2,x)\Big] + \lim_{n \to \infty}\left[nB_{(n,0,\alpha,\beta)}(R(t,x)(t-x)^2.x)\right]$$

$$= [1 + 2x + \alpha - \beta x]f^{(1)}(x) + \left[\frac{x^2+2x}{2}\right]f^{(2)}(x)$$

We can also see that from lemma 2.3,

$$B_{(n,0,\alpha,\beta)}(t^2)(x) = \frac{x^2n^4 + (4x+2\alpha x)n^3 + (-6\alpha x + \alpha^2 + 2\alpha + 2)n^2 + (-5\alpha^2 - 6\alpha)n + 6\alpha^2}{(n-2)(n-3)(n+\beta)^2}$$

And hence,

$$\lim_{n \to \infty} n\left(B_{(n,0,\alpha,\beta)}(t^2)(x) - x^2\right) = x(4 + 5x + 2\alpha - 2x\beta).$$

4. CONCLUSION

It is noted that our stancu generalized operators $(B_{(n,\alpha,\beta)}f)(x)$ gives better results in simultaneous approximation. From (2.3) and (2.4), we observe that,

$$T_{(r,n,m,\alpha,\beta)}(x) = O\left(n^{-[(m+1)/2]}\right)$$

We see from Theorem 3.1, that our operator has improved result for $f^{(1)}(x)$, $f^{(2)}(x)$.

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