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A COMMON FIXED POINT THEOREM FOR QUADRUPLE OF SELF MAPPINGS SATISFYING WEAK CONDITIONS AND APPLICATIONS

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Abstract: In this paper, we establish a common fixed point theorem for a quadruple of self mappings satisfying a common (E.A) property on a metric space satisfying weakly compatibility and a generalized Φ-contraction. Since we consider several weak conditions, our results improve and extend several known results. Further, we justify our results with suitable example and provide application.

1 Introduction and Preliminaries

In mathematics, a fixed point (also known as an invariant point) of a function is a point that is mapped to itself by the function. In many fields, equilibria or stability are fundamental concepts that can be described in terms of fixed points. Many applications of fixed point theorems can be found both on the theoretical side and on the applied side. Logician Saul Kripke makes use of fixed points in his influential theory of truth. In the theory of Phase Transitions, linearisation near an unstable fixed point has led to Wilson's Nobel prize-winning work inventing the renormalization group, and to the mathematical explanation of the term "critical phenomenon". The vector of Page Rank values of all web pages is the fixed point of a linear transformation derived from the www's link structure.

Two self mappings A and S of a metric space (X, d) are called compatible if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some t in X. In 1986, the notion of compatible mappings which generalized commuting mappings, was introduced by Jungck [3]. This has proven useful for generalization of results in metric fixed point theory for single-valued as well as multi-valued mappings. Further in 1998, the more general class of mappings called weakly compatible mappings was introduced by Jungck and Rhoades [4]. Recall that self mappings S and T of a metric space (X, d) are called weakly compatible if $Sx = Tx$ for some $x \in X$ implies that $STx = TSx$. i.e., weakly compatible pair commute at coincidence points.

Recently Aamri and Moutawakil [1] introduced the following notion for a pair of maps as :

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Definition 1.1 *Let S and T be two self mappings of a metric space* (X, d) *. Then S and T are said to satisfy the property* $(E.A)$ *, if there exists a sequence* $\{x_n\}$ *in X such that* $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$*, for some* $t \in X$ *.*

Recently, Liu, Wu and Liu [5] defined a common property (E.A) for pairs of mappings as follows:

Definition 1.2 *Let* $A, B, S, T : X \to X$ *. The pairs* (A, S) *and* (B, T) *are said to satisfy a common property* $(E.A)$ *, if there exist two sequences* $\{x_n\}$ *and* $\{y_n\}$ *such that* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t \in X$ *.*

If $B = A$ and $S = T$ in the above Definition 1.2, we have the Definition 1.1. In order to justify the Definition 1.2, we give the following example.

Example 1.3 *Consider the usual metric d on the closed unit interval* $X = [0,1]$*. Let A, B, S and T be self mappings on X, defined by*

$$Ax = 1 - \frac{x}{2} \text{ when } x \in [0, \frac{1}{2}),$$
$$= 1, \text{ when } x \in [\frac{1}{2}, 1],$$
$$Sx = 1 - 2x \text{ when } x \in [0, \frac{1}{2}),$$
$$= 1, \text{ when } x \in [\frac{1}{2}, 1],$$

$Bx = 1 - x$ and $Tx = 1 - \frac{x}{3}$, for all $x \in X$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X, defined by $x_n = \frac{1}{n+1}$ and $y_n = \frac{1}{n^2+1}$, for all $n \in N$. Then $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 1 \in X$.

Throughout the paper, we denote by Φ the collection of all functions $\varphi : [0,\infty) \to [0,\infty)$ which are upper semi-continuous from the right, non-decreasing and satisfy $\lim_{s\to t+} \sup \varphi(s) < t$, $\varphi(t) < t$ for all $t > 0$.

Let A, B, S and T be self-mappings on a metric space X endowed with metric d such that

$$[d^p(Ax,By)+ad^p(Sx,Ty)]d^p(Ax,By)$$

$$\leq a\max\{d^p(Ax,Sx)d^p(By,Ty),d^q(Ax,Ty)d^{q'}(By,Sx)\}$$

$$+\max\{\varphi_1(d^{2p}(Sx,Ty)),\varphi_2(d^r(Ax,Sx)d^{r'}(By,Ty)),\varphi_3(d^s(Ax,Ty)d^{s'}(By,Sx)),$$

$$\varphi_4(\frac{1}{2}[d^l(Ax,Ty)d^{l'}(Ax,Sx)+d^l(By,Sx)d^{l'}(By,Ty)])\}, \quad (1.1)$$

for all $x,y\in X$, $\varphi_i\in\Phi(i=1,2,3,4)$, $a,p,q,q',r,r',s,s',l,l'\geq 0$ and $2p=q+q' = r+r' = s+s' = l+l'$. The condition (1.1) is commonly called a generalized Φ - contraction.

In main aim of this paper is to prove some common fixed point theorems for a quadruple of weak compatible self mappings on a metric space satisfying a common (E.A) property and a generalized Φ - contraction. These theorems extend and generalize many results including the Pathak et al. [7, 8].

2 Main Results

In this section, we give the main results of this paper.

Theorem 2.1 *Let A, B, S and T be self mappings on a metric space* (X,d) *satisfying* (1.1)*. If the pairs* (A,S) *and* (B,T) *satisfy a common* $(E.A)$ *property, weakly compatible and that* $T(X)$ *and* $S(X)$ *are closed subsets of X, then A, B, S and T have a unique common fixed point in X.*

Proof. Since (A,S) and (B,T) satisfy a common property (E.A), there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\lim_{n\to\infty}Ax_n=\lim_{n\to\infty}Sx_n=\lim_{n\to\infty}By_n=\lim_{n\to\infty}Ty_n=z,$$

for some $z\in X$. Assume that $S(X)$ and $T(X)$ are closed subspaces of X. Then, $z=Su=Tv$ for some $u,v\in X$. Now using (1.1) by considering x in place of x_n and v in place of y, we have

$$[d^p(Ax_n,Bv)+ad^p(Sx_n,Tv)]d^p(Ax_n,Bv)\leq a\max\{d^p(Ax_n,Sx_n)d^p(Bv,Tv),$$

$$d^{q}(Ax_n,Tv)d^{q'}(Bv,Sx_n)\}+\max\{\varphi_1(d^{2p}(Sx_n,Tv)),$$
$$\varphi_2(d^{r}(Ax_n,Sx_n)d^{r'}(Bv,Tv)),\varphi_3(d^{s}(Ax_n,Tv)d^{s'}(Bv,Sx_n)),$$
$$\varphi_4(\frac{1}{2}[d^{l}(Ax_n,Tv)d^{l'}(Ax_n,Sx_n)+d^{l}(Bv,Sx_n)d^{l'}(Bv,Tv)])\}.$$

Taking limit as $n\to\infty$, and $Tv=z$, we obtain

$$[d^{p}(z,Bv)+ad^{p}(z,z)]d^{p}(z,Bv)\le a\max\{d^{p}(z,z)d^{p}(Bv,z),$$
$$d^{q}(z,z)d^{q'}(Bv,z)\}+\max\{\varphi_1(d^{2p}(z,z)),$$
$$\varphi_2(d^{r}(z,z)d^{r'}(Bv,z)),\varphi_3(d^{s}(z,z)d^{s'}(Bv,z)),$$
$$\varphi_4(\frac{1}{2}[d^{l}(z,z)d^{l'}(z,z)+d^{l}(Bv,z)d^{l'}(Bv,z)])\}$$

or $$d^{2p}(z,Bv)\le\max\{\varphi_1(0),\varphi_2(0),\varphi_3(0),\varphi_4(\frac{1}{2}d^{l+l'}(Bv,z))\},$$

or $$d^{2p}(z,Bv)\le\max\{\varphi_1(d^{2p}(z,Bv)),\varphi_2(d^{r+r'}(z,Bv)),$$
$$\varphi_3(d^{s+s'}(z,Bv)),\varphi_4(\frac{1}{2}d^{l+l'}(Bv,z))\}.$$

This together with a well known result of Chang [2] which states that if $\varphi_i\in\Phi$ where $i\in I$ (some index set), then there exists a $\varphi\in\Phi$ such that $\max\{\varphi_i,i\in I\}\le\varphi(t)$ for all $t>0$ and the property $\varphi(t)<t$ for all $t>0$; we have

$$d^{2p}(z,Bv)\le\varphi(d^{2p}(z,Bv))<d^{2p}(z,Bv),$$

a contradiction. This implies that $z=Bv$. Therefore $Tv=z=Bv$. Hence it follows by the weak compatibility of the pair (B,T) that $BTv=TBv$, that is $Bz=Tz$.

Now, we shall show that z is a common fixed point of B and T. For this, considering x_n in place of x and z in place of y, (1.1) becomes

$$[d^{p}(Ax_n,Bz)+ad^{p}(Sx_n,Tz)]d^{p}(Ax_n,Bz)\le a\max\{d^{p}(Ax_n,Sx_n)d^{p}(Bz,Tz),$$
$$d^{q}(Ax_n,Tz)d^{q'}(Bz,Sx_n)\}+\max\{\varphi_1(d^{2p}(Sx_n,Tz)),$$
$$\varphi_2(d^{r}(Ax_n,Sx_n)d^{r'}(Bz,Tz)),\varphi_3(d^{s}(Ax_n,Tz)d^{s'}(Bz,Sx_n)),$$
$$\varphi_4(\frac{1}{2}[d^{l}(Ax_n,Tz)d^{l'}(Ax_n,Sx_n)+d^{l}(Bz,Sx_n)d^{l'}(Bz,Tz)])\}$$

Letting $n\to\infty$, with the help of the fact that $\lim_{n\to\infty}Ax_n = z = \lim_{n\to\infty}Sx_n$ and $Bz=Tz$, we get

$$[d^{p}(z,Bz)+ad^{p}(z,Tz)]d^{p}(z,Bz)\le a\max\{d^{p}(z,z)d^{p}(Bz,Tz),$$
$$d^{q}(z,Tz)d^{q'}(Bz,z)\}+\max\{\varphi_1(d^{2p}(z,Tz)),$$

$\varphi_2(d^r(z,z)d^{r'}(Bz,Tz)),\varphi_3(d^s(z,Tz)d^{s'}(Bz,z)),$

$\varphi_4(\frac{1}{2}[d^l(z,Tz)d^{l'}(z,z)+d^l(Bz,z)d^{l'}(Bz,Tz)])\}$,

or $\quad d^{2p}(z,Bz)+ad^{2p}(z,Bz)\le ad^{q+q'}(Bz,z)+\max\{\varphi_1(d^{2p}(z,Bz)),$

$\varphi_2(0),\varphi_3(d^{s+s'}(z,Bz)),\varphi_4(0)\},$

or $\quad (1+a)d^{2p}(z,Bz)\le ad^{q+q'}(Bz,z)+\max\{\varphi_1(d^{2p}(z,Bz)),$

$\varphi_2(0),\varphi_3(d^{s+s'}(z,Bz)),\varphi_4(0)\},$

or $\quad d^{2p}(z,Bz)\le \frac{a}{1+a}d^{q+q'}(Bz,z)+\frac{1}{1+a}\max\{\varphi_1(d^{2p}(z,Bz)),$

$\varphi_2(0),\varphi_3(d^{s+s'}(z,Bz)),\varphi_4(0)\}$

Since $2p=q+q'$, using the same argument as applied in the previous similar part, we have

$$d^{2p}(z,Bz)<d^{2p}(z,Bz),$$

a contradiction. So, $z=Bz=Tz$. Thus z is a common fixed point of B and T.

Similarly, we can prove that z is a common fixed point of A and S. Thus z is the common fixed point of A, B, S and T. The uniqueness of z as a common fixed point of A, B, S and T can easily be verified.

In Theorem 2.1, if we put $a=0$ and $\varphi_i(t)=ht$ $(i=1,2,3,4)$, where $0<h<1$, we get the following corollary:

Corollary 2.2 *Let A, B, S and T be self mappings of a metric space X. If the pairs (A,S) and (B,T) satisfy a common (E.A) property and*

$$d^{2p}(Ax,By)\le h\max\{d^{2p}(Sx,Ty),d^r(Ax,Sx)d^{r'}(By,Ty),d^s(Ax,Ty)$$

$$d^{s'}(By,Sx),\frac{1}{2}[d^l(Ax,Ty)d^{l'}(Ax,Sx)+d^l(By,Sx)d^{l'}(By,Ty)]\},\qquad(2.1)$$

for all $x,y\in X$, $p,q,q',r,r',s,s',l,l'\ge 0$ and $2p=q+q'=r+r'=s+s'=l+l'$. If the pairs (A,S) and (B,T) are weakly compatible and that $T(X)$ and $S(X)$ are closed, then A, B, S and T have a unique common fixed point in X.

Especially, when

$\max\{d^{2p}(Sx,Ty),d^r(Ax,Sx)d^{r'}(By,Ty),d^s(Ax,Ty)d^{s'}(By,Sx),$

$\frac{1}{2}[d^{l}(Ax,Ty)d^{l'}(Ax,Sx)+d^{l}(By,Sx)d^{l'}(By,Ty)]\} = d^{2p}(Sx,Ty)$, it generalizes Corollary 3.9 of Pathak et al. [8].

In Theorem 2.1, if we take $S=T=I_X$ (the identity mapping on X), then we have the following corollary:

Corollary 2.3 *Let A and B be self mappings of a complete metric space X satisfying the following condition:*

$$[d^{p}(Ax,By)+ad^{p}(x,y)]d^{p}(Ax,By) \le a\max\{d^{p}(Ax,x)d^{p}(By,y),\ d^{q}(Ax,y)\, d^{q'}(By,x)\} + \max\{\varphi_1(d^{2p}(x,y)),\ \varphi_2(d^{r}(Ax,x)d^{r'}(By,y)), \varphi_3(d^{s}(Ax,y)d^{s'}(By,x)),\ \varphi_4(\tfrac{1}{2}[d^{l}(Ax,y)d^{l'}(Ax,x)+d^{l}(By,x)d^{l'}(By,y)])\},$$

for all $x,y\in X$, $\varphi_i\in\Phi$ $(i=1,2,3,4)$, $a,p,q,q',r,r',s,s',l,l'\ge 0$ and $2p=q+q' = r+r' = s+s' = l+l'$, then A and B have a unique common fixed point in X.

As an immediate consequence of Theorem 2.1 with $S=T$, we have the following:

Corollary 2.4 *Let A, B and S be self-mappings of X such that (A,S) and (B,S) satisfy a common $(E.A)$ property and*

$$d^{2p}(Ax,By) \le a\max\{d^{p}(Ax,Sx)d^{p}(By,Sy),\ d^{q}(Ax,Sy)\ d^{q'}(By,Sx)\} + \max\{\varphi_2(d^{r}(Ax,Sx)d^{r'}(By,Sy)),\ \varphi_3(d^{s}(Ax,Sy)d^{s'}(By,Sx)), \varphi_4(\tfrac{1}{2}[d^{l}(Ax,Sy)d^{l'}(Ax,Sx)+d^{l}(By,Sx)d^{l'}(By,Sy)])\}, \quad (2.2)$$

for all $x,y\in X$, $\varphi_i\in\Phi$ $(i=2,3,4)$, $a,p,q,q',r,r',s,s',l,l'\ge 0$ and $2p=q+q' = r+r' = s+s' = l+l'$. If the pairs (A,S) and (B,S) are weakly compatible and that $S(X)$ is closed, then A, B and S have a unique common fixed point in X.

Theorem 2.5 *Let S, T and A_n $(n\in N)$ be self mappings of a metric space (X,d). Suppose further that the pairs (A_{2n-1},S) and (A_{2n},T) are weakly compatible for any $n\in N$ and satisfying a common $(E.A)$ property. If $S(X)$ and $T(X)$ are closed and that for any $i\in N$, the following condition is satisfied for all $x,y\in X$*

$$[d^{p}(A_ix,A_{i+1}y)+ad^{p}(Sx,Ty)]d^{p}(A_ix,A_{i+1}y)\le a\max\{d^{p}(A_ix,Sx)\, d^{p}(A_{i+1}y,Ty), d^{q}(A_ix,Ty)d^{q'}(A_{i+1}y,Sx)\}+\max\{\varphi_1(d^{2p}(Sx,Ty)),$$

$$\varphi_2(d^r(A_i x, Sx)d^{r'}(A_{i+1}y, Ty)), \varphi_3(d^s(A_i x, Ty)d^{s'}(A_{i+1}y, Sx)),$$
$$\varphi_4(\frac{1}{2}[d^l(A_i x, Ty)d^{l'}(A_i x, Sx) + d^l(A_{i+1}y, Sx)d^{l'}(A_{i+1}y, Ty)])\},$$

where $\varphi_i \in \Phi$ $(i = 1,2,3,4)$, $a, p, q, q', r, r', s, s', l, l' \geq 0$ and $2p = q + q' = r + r' = s + s' = l + l'$, then S, T and A_n $(n \in N)$ have a common fixed point in X.

3. Example and Application

Let us consider the functions $\varphi_i : [0,\infty) \to [0,\infty)$ defined by

$$\varphi_i(x) = \frac{1}{1+ix}, \text{ for all } x \in [0,\infty) \text{ and } i = 1, 2, 3, 4.$$

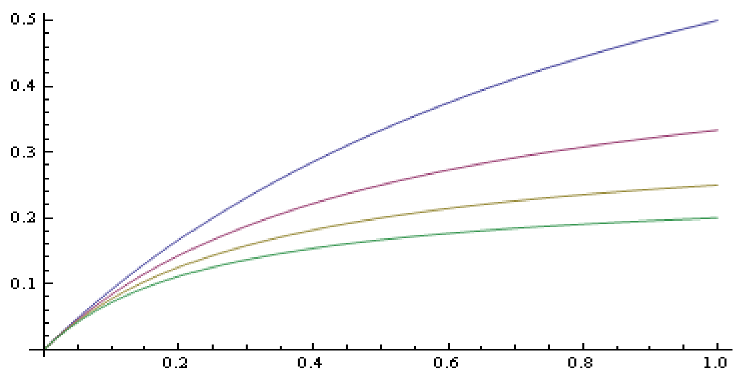

Fig 1. Graph of φ_i 's

From the fig 1, it is clear that each φ_i satisfy the required condition. Now let us consider the metric space *X* = [0, 1] under usual metric. Also consider the self mappings on *X* as follows:

$A(x) = x^3$, $S(x) = x^2$ and $B(x) = 1$, $T(x) = 2$-x, for all *x* in *X*.

Then the pairs (*A*, *S*) and (*B*, *T*) are not compatible, because for the sequence $\{x_n\}$, defined by $x_n = n^{-4}$ (for all *n* in *N*), the condition of compatibility is not satisfied. But they are weakly compatible, since they commute at coincidence points. Further, it is easy to see that both the pairs satisfy a common property (E.A) and generalized Φ - contraction.

Then the main theorem of this paper guarantees a unique common fixed point in *X*, which is '1'. Now we will see it graphically as follows:

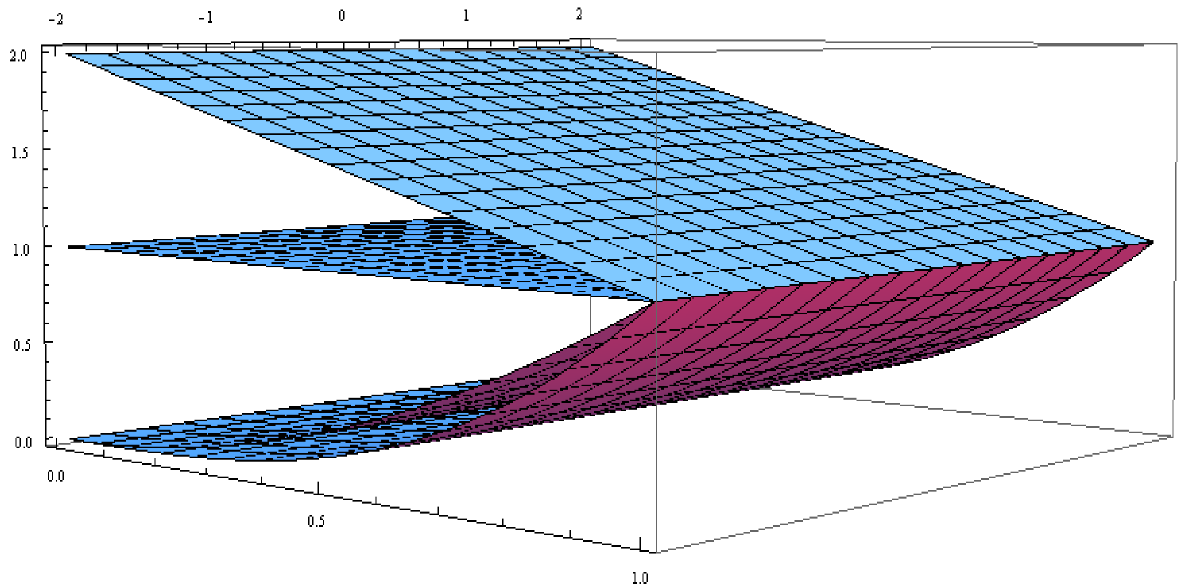

Fig 2. Unique fixed point '1' is shown graphically in 3D for the pairs (A, S) and (B, T).

From the figure, we observe that by means of certain transformations through a fixed point we can have different structures from original structure or vice-versa. The results of the paper can be applied to other branch of applied sciences with this aim. The results or their extended form (which of course need further research with this specific aim) may also be used to construct fixed points in Euclidean geometry, which generally require the use of a compass and ruler. These can be achieved by replacing geometrical figures by suitable (approximate) functions.

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