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# ON THE GEOMETRY OF HEMI-SLANT SUBMANIFOLDS OF LP-COSYMPLECTIC MANIFOLD

SIDDESHA M.S.[1,*], BAGEWADI C.S.[2] AND VENKATESHA S.[2]

[1]Department of mathematics, New Horizon College of Engineering, Bangalore, India

[2]Department of Mathematics, Kuvempu University, Shankaraghatta, Shimoga, Karnataka, India

[*]Correspondence: mssiddesha@gmail.com

Abstract. In this paper, we study hemi-slant submanifolds of LP-cosymplectic manifold. We investigate the integrability conditions of the distributions of hemi-slant submanifolds and also obtain some interesting results.

## 1. Introduction

The geometry of slant submanifolds has become an interesting field since introduction of the notion of slant submanifolds of an almost Hermitian manifolds and these are the generalization of both holomorphic and totally real submanifolds. The contact version of such submanifolds were defined and studied by Lotta [15]. Later Cabrerizo et. al., [2] investigated slant submanifolds of Sasakian manifold. Further such submanifolds of different ambient spaces have been studied by many geometers (See. [8, 11, 21, 24, 25]).

*Key words and phrases.* Slant submanifold, hemi-slant submanifold and LP-cosymplectic manifold.

In [18], Papaghiuc defined semi-slant submanifolds, as a generalization of slant and CR submanifolds. Many geometers have studied such type of submanifolds of different known spaces (See. [3, 11, 22] and references therein). As a natural generalization of semi-slant submanifolds, Carriazo [4] introduced the notion of bi-slant submanifolds and hemi-slant submanifold is the particular case of bi-slant submanifolds. In the beginning, Carriazo initiated the study of hemislant submanifolds in the name of anti-slant submanifolds. But it seems that the name anti-slant has no slant factor, so Sahin [20] gave the name of hemi-slant submanifolds instead of anti-slant submanifolds. The study of such submanifolds of different ambient manifolds were studied by many geometers of [1, 10, 13, 14, 23].

Recently, Siraj Uddin et. al., [25] studied totally umbilical hemi-slant submanifolds of LP-cosymplectic manifold and proved the characterization theorem. In the present paper, we continue the work of [25] and obtain some interesting results.
The paper is organized as follows: In section-2, we recall the notion of LP-cosymplectic manifold and some basic results of submanifolds, which are used for further study. Section-3 is devoted to study hemi-slant submanifolds of LP-cosymplectic manifold. We obtain the integrability conditions of the distributions which are involved in the definition of hemi-slant submanifolds and obtain some interesting results.

## 2. **Preliminaries**

2.1. **LP-Cosymplectic manifold.** Let $\tilde{M}$ be a $(2m+1)$-dimensional smooth connected paracompact Hausdorff manifold equipped with a Lorentzian metric $g$, that is, $g$ is a smooth symmetric tensor field of type $(0,2)$ such that at every point $p \in \tilde{M}$, the tensor
$g_p : T_p\tilde{M} \otimes T_p\tilde{M} \to R$ is a non-degenerate inner product of signature, $(-,+,\dots,+)$ where $T_p\tilde{M}$ is the tangent space of $\tilde{M}$ at $p$ and $R$ is the real line. In other words, a matrix representation of $g_p$ has one eigenvalue negative and all other eigenvalues positive. Then $\tilde{M}$ is Lorentzian manifold. A non-zero vector $X_p \in T_p\tilde{M}$ is known to be spacelike, null, non-spacelike or timelike according as $g_p(X_p, X_p) > 0,\ = 0,\ \leq 0$ or $< 0$ respectively.

Let $\tilde{M}$ be equipped with a triple $(\phi, \xi, \eta)$, where $\phi$ is a (1,1) tensor field, $\xi$ is a vector field, $\eta$ is a 1-form on $\tilde{M}$ such that

$$\phi^2 = I + \eta \otimes \xi, \qquad \eta(\xi) = -1, \tag{2.1}$$

$$\eta \circ \phi = 0, \quad \phi\xi = 0, \qquad rank(\phi) = 2n. \tag{2.2}$$

If $\tilde{M}$ admits a Lorentzian metric $g$, such that

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

then $\tilde{M}$ is said to admit a Lorentzian almost paracontact structure $(\phi, \xi, \eta, g)$. In this case, we get

$$g(X, \xi) = \eta(X), \tag{2.4}$$

$$\Phi(X, Y) = g(\phi X, Y) = g(X, \phi Y) = \Phi(Y, X) \tag{2.5}$$

$$(\tilde{\nabla}_X \phi)Y = g(X, Y)\xi + 2\eta(X)\eta(Y)\xi, \tag{2.6}$$

where $\tilde{\nabla}$ is the covariant differentiation with respect to $g$. The Lorentzian metric $g$ makes $\xi$ a timelike unit vector field, that is, $g(\xi, \xi) = -1$ (see [16, 17]).

A Lorentzian almost paracontact manifold is called a LP-cosymplectic manifold [19] if

$$(\tilde{\nabla}_X \phi)Y = 0, \quad \tilde{\nabla}_X \xi = 0. \tag{2.7}$$

2.2. **Submanifolds.** Throughout the paper, we denote $\tilde{M}$ as LP-cosmplectic manifold, $M$ as a submanifold of $\tilde{M}$ and $\xi$ is structure vector field tangent to $M$. If $i : M \to \tilde{M}$ ia an isometric immersion then the Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{2.8}$$

$$\tilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.9}$$

for any $X, Y \in TM$, $V \in T^{\perp}M$, where $\nabla$ (resp. $\nabla^{\perp}$) is the induced connection on the tangent bundle $TM$ (resp. normal bundle $T^{\perp}M$). The shape operator $A$ is related to the second fundamental form $\sigma$ of $M$ by

$$g(A_V X, Y) = g(\sigma(X, Y), V). \tag{2.10}$$

A submanifold $M$ of dimension $(2n+1)$ of a Lorentzian almost paracontact manifold $\tilde{M}$ is said to be totally umbilical if

$$\sigma(X,Y) = g(X,Y)H, \tag{2.11}$$

where $H$ is the mean curvature vector and is defined by

$$H = \frac{1}{2n+1}trace(\sigma) = \frac{1}{2n+1}\sum_{i=1}^{2n+1}\sigma(e_i,e_i), \tag{2.12}$$

here $\{e_1, e_2, \ldots, e_{2n+1}\}$ is the local orthonormal frame of $M$.
For any $X \in \Gamma(TM)$ and for any $V \in \Gamma(T^{\perp}M)$, we can write

$$\phi X = TX + NX, \quad \phi V = tV + nV, \tag{2.13}$$

where $TX$ (resp. $tV$) and $NX$ (resp. $nV$) are the tangential and normal component of $\phi X$ (resp. $\phi V$). The covariant derivatives of the tensor fields $T$, $N$, $t$ and $n$ are defined as

$$(\nabla_X T)Y = \nabla_X(TY) - T(\nabla_X Y),\ \forall X,Y \in TM, \tag{2.14}$$

$$(\nabla_X N)Y = \nabla^{\perp}_X(NY) - N(\nabla_X Y),\ \forall X,Y \in TM, \tag{2.15}$$

$$(\nabla_X t)V = \nabla_X(tV) - t(\nabla_X V),\ \forall X \in TM,\ V \in T^{\perp}M, \tag{2.16}$$

$$(\nabla_X n)V = \nabla_X(nV) - n(\nabla_X V),\ \forall X \in TM,\ V \in T^{\perp}M. \tag{2.17}$$

2.3. **Slant submaifolds.** In the present section, we consider $M$ is a proper slant submanifold of a LP-cosymplectic manifold $\tilde{M}$. We always consider such submanifolds tangent to the structure vector fields $\xi$.
An immersed submanifold $M$ of a Lorentzian paracontact manifold $\tilde{M}$ is slant in $\tilde{M}$ if for any $x \in M$ and any $X \in T_xM$ such that $X, \xi$ are linearly independent, the angle $\theta(x) \in [0, \frac{\pi}{2}]$ between $\phi X$ and $T_xM$ is a constant $\theta$, i.e., $\theta$ does not depend on the choice of $X$ and $x \in M$, $\theta$ is called the slant angle of $M$ in $\tilde{M}$. Invariant and anti-invariant submanifolds are slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively [9].
We have the following theorem which characterize slant submanifolds of a Lorentzian paracontact manifold

**Theorem 2.1.** [9] *Let $M$ be a submanifold of a Lorentzian paracontact manifold $\tilde{M}$ such that $\xi \in TM$. Then, $M$ is slant if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$T^2 = \lambda(I + \eta \otimes \xi). \tag{2.18}$$

*Further more, if $\theta$ is the slant angle of $M$, then $\lambda = cos^2\theta$.*

From [9], for any $X, Y$ tangent to $M$, we can easily obtain the results for a Lorentzian paracontact manifold $\tilde{M}$,

$$g(TX, TY) = cos^2\theta\{g(X,Y) + \eta(X)\eta(Y)\} \tag{2.19}$$

$$g(NX, NY) = sin^2\theta\{g(X,Y) + \eta(X)\eta(Y)\} \tag{2.20}$$

## 3. **Hemi-slant submanifolds of LP-cosymplectic manifold**

In the present section, we verify the integrability of the distributions involved in the definition of hemi-slant submanifolds of LP-cosymplectic manifold and also obtain some interesting results.

**Definition 3.1.** *A submanifold $M$ of $\tilde{M}$ is said to be hemi-slant submanifold of an almost contact metric manifold $\tilde{M}$ if there exist two orthogonal complementary distribution $D^\theta$ and $D^\perp$ on $M$ such that*
*(i) $TM = D^\theta \oplus D^\perp \oplus < \xi >$;*
*(ii) the distribution $D^\theta$ is slant with slant angle $\theta \neq \frac{\pi}{2}$;*
*(iii) the distribution $D^\perp$ is an anti-invariant i.e., $\phi D^\perp \subseteq T^\perp M$.*

It is clear from above that CR-submanifolds and slant submanifolds are hemi-slant submanifolds with slant angle $\theta = \frac{\pi}{2}$ and $D^\theta = 0$, respectively.
Let $M$ be a hemi-slant submanifold of an almost contact metric manifold $\tilde{M}$, and $X \in TM$. Then as $TM = D^\theta \oplus D^\perp \oplus < \xi >$, we write

$$X = P_1X + P_2X + \eta(X)\xi, \tag{3.1}$$

where $P_1X \in D^\theta$ and $P_2X \in D^\perp$. Now by equations (2.13) and (3.1)

$$\phi X = TP_1X + NP_1X + \phi P_2X.$$

It is easy to see that

$$\phi P_2X = NP_2X,\ TP_2X = 0,\ TP_1X \in D^\theta.$$

Thus

$$TX = TP_1X,\ NX = NP_1X + NP_2X.$$

By simple computation, we can state the following lemma:

**Lemma 3.1.** *For a hemi-slant submanifold $M$ of a LP-cosymplectic manifold $\tilde{M}$ and for all $X, Y \in TM$, $V \in T^\perp M$, we have*

$$\nabla_X TY - A_{NY}X - T\nabla_X Y + t\sigma(X,Y) = 0, \tag{3.2}$$

$$\sigma(X,TY) + \nabla^\perp_X NY - N\nabla_X Y + n\sigma(X,Y) = 0, \tag{3.3}$$

$$\nabla_X tV - A_{nV}X + TA_VX - t\nabla^\perp_X V = 0, \tag{3.4}$$

$$\sigma(X,tV) + \nabla^\perp_X nV + NA_VX - n\nabla^\perp_X V = 0. \tag{3.5}$$

**Lemma 3.2.** *If $M$ is a hemi-slant submanifold of a LP-cosymplectic manifold $\tilde{M}$ such that $\xi \in TM$, then*

$$\sigma(X,\xi) = 0, \qquad \sigma(TX,\xi) = 0, \qquad \nabla_X \xi = 0, \tag{3.6}$$

*for all $X \in TM$.*

*Proof.* The result directly follow from the use of (2.7) and (2.8). □

**Theorem 3.2.** *For a hemi-slant submanifold $M$ of a LP-cosymplectic manifold $\tilde{M}$ and for all $X, Y \in D^\theta$, we have*

$$A_{\phi X}Y = A_{\phi Y}X.$$

*Proof.* By virtue of (2.10), we have

$$\begin{aligned} g(A_{\phi X}Y, Z) &= g(\sigma(Y,Z), \phi X) \\ &= g(\phi\sigma(Y,Z), X) \\ &= g(\phi\tilde{\nabla}_Z Y, X) - g(\phi\nabla_Z Y, X) \\ &= g(\phi\tilde{\nabla}_Z Y, X). \end{aligned}$$

Using (2.7) and (2.10), we have

$$g(A_{\phi X}Y, Z) = g(A_{\phi Y}X, Z).$$

Hence the result. □

In [25], authors found the necessary condition for a distribution $D^{\perp}$ to be integrable and it is stated as follows:

**Theorem 3.3.** [25] *Let $M$ be a hemi-slant submanifold $M$ of a LP-cosymplectic manifold $\tilde{M}$. Then the distribution $D^{\perp}$ is integrable if and only if*

$$A_{NZ}W = A_{NW}Z, \tag{3.7}$$

*for all $Z, W \in D^{\perp}$. Now, we find the necessary conditon for a slant distribution $D^{\theta}$ and $D^{\theta} \oplus D^{\perp}$ to be integrable:*

**Theorem 3.4.** *Let $M$ be a hemi-slant submanifold $M$ of a LP-cosymplectic manifold $\tilde{M}$. Then the distribution $D^{\perp} \oplus D^{\theta}$ is always integrable.*

*Proof.* For $X, Y \in D^{\perp} \oplus D^{\theta}$, we have

$$\begin{aligned} g([X,Y],\xi) &= g(\nabla_X Y, \xi) - g(\nabla_Y X, \xi) \\ &= g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X) \end{aligned}$$

By virtue of (3.6), we have

$$g([X,Y],\xi) = 0.$$

□

**Theorem 3.5.** *Let $M$ be a hemi-slant submanifold $M$ of a LP-cosymplectic manifold $\tilde{M}$. Then the distribution $D^{\theta}$ is integrable if and only if*

$$\sigma(X, TY) - \sigma(Y, TX) + \nabla^{\perp}_X NY - \nabla^{\perp}_Y NX \in \mu \oplus N(D^{\theta}),$$

*for all $X, Y \in D^{\theta}$.*

*Proof.* For $X, Y \in D^\theta$ and $Z \in D^\perp$, we have

$$g([X,Y],Z) = g(\tilde{\nabla}_X Y - \tilde{\nabla}_Y X, Z).$$

By virtue of (2.1), (2.2) and (2.3), the above equation yields

$$g([X,Y],Z) = g(\phi\tilde{\nabla}_X Y, \phi Z) - g(\phi\tilde{\nabla}_Y X, \phi Z).$$

Further, by applying (2.8) and (2.9) we obtain

$$g([X,Y],Z) = g(\sigma(X,TY) - \sigma(Y,TX) + \nabla^\perp_X NY - \nabla^\perp_Y NX, \phi Z).$$

As $\phi Z \in \phi(D^\perp)$ and $N(D^\theta)$ and $N(D^\perp)$ are orthogonal to each other in the normal bundle $T^\perp M$, thus the result follows. □

**Theorem 3.6.** *Let $M$ be a hemi-slant submanifold $M$ of a LP-cosymplectic manifold $\tilde{M}$. Then the distribution $D^\theta$ is integrable if and only if*

$$P_2\{\nabla_X TY - \nabla_Y TX + A_{NX}Y - A_{NY}X\} = 0,$$

*for all $X, Y \in D^\theta$.*

*Proof.* Let $P_1$ and $P_2$ denote the projections on $D^\theta$ and $D^\perp$ respectively. From (2.7), for any $X, Y \in D^\theta$ we have

$$(\tilde{\nabla}_X \phi)Y = 0.$$

Using equations (2.8), (2.9) and (2.13), we obtain

$$\nabla_X TY + \sigma(X,TY) - A_{NY}X + \nabla^\perp_X NY - T\nabla_X Y - N\nabla_X Y - t\sigma(X,Y) - n\sigma(X,Y) = 0.$$

Comparing the tangential components, we have

$$\nabla_X TY - A_{NY}X - T\nabla_X Y - t\sigma(X,Y) = 0. \tag{3.8}$$

Replacing $X$ and $Y$ and subtract the obtained result from (3.8), we get

$$T[X,Y] = \nabla_X TY - \nabla_Y TX + A_{NX}Y - A_{NY}X. \tag{3.9}$$

Applying $P_2$ to (3.9), we obtain the result. □

**Theorem 3.7.** *Let $M$ be a hemi-slant submanifold $M$ of a LP-cosymplectic manifold $\tilde{M}$. If the leaves of $D^\theta$ are totally geodesic in $M$, then*

$$\nabla_X \phi Y = \phi\sigma(X, Y),$$

*for each $X, Y \in D^\theta$ and $Z, W \in D^\perp$.*

*Proof.* From (2.7), for any $X, Y \in D^\theta$ we have

$$(\tilde{\nabla}_X \phi)Y = 0.$$

For $Z \in D^\perp$, using equations (2.8), (2.9) and (2.13), we get

$$g(\nabla_X \phi Y, Z) - g(\phi\nabla_X Y, Z) = g(\sigma(X, Y), \phi Z).$$

Therefore from the above equation, we obtain the result. □

**Theorem 3.8.** *Let $M$ be a hemi-slant submanifold $M$ of a LP-cosymplectic manifold $\tilde{M}$. If the leaves of $D^\perp$ are totally geodesic in $M$, then*

$$A_{NW} Z = \nabla_Z TW,$$

*for $X, \in D^\theta$ and $Z, W \in D^\perp$.*

*Proof.* From (2.7), for any $Z, W \in D^\perp$ we have

$$(\tilde{\nabla}_Z \phi)W = 0.$$

Using equations (2.8), (2.9) and (2.13), we get

$$\nabla_Z TW + \sigma(Z, TW) - A_{NW} Z + \nabla^\perp_Z NW = \phi\nabla_Z W + \phi(Z, W).$$

For $X \in D^\theta$, we have

$$g(\nabla_Z TW, X) - g(A_{NW} Z, X) = g(\phi\nabla_Z W, X).$$

Therefore from the above equation, we have

$$g(\nabla_Z W, \phi X) = g(\nabla_Z TW - A_{NW} Z, X).$$

The leaves of $D^\perp$ are totally geodesic in $M$, if for $Z, W \in D^\perp$, $\nabla_Z W \in D^\perp$. Therefore the result follows from the above equation. □

**Theorem 3.9.** *Let $M$ be a hemi-slant submanifold of a LP-cosymplectic manifold $\tilde{M}$. If the leaves of $D^\theta$ are totally geodesic in $M$, then*

$$\nabla_X \phi Y = \phi\sigma(X, Y),$$

*for all $X, Y \in D^\theta$ and $Z \in D^\perp$.*

*Proof.* From (2.7), for any $X, Y \in D^\theta$ we have

$$(\tilde{\nabla}_X \phi)Y = 0.$$

Now for $Z \in D^\perp$ and using equations (2.8), (2.9) and (2.13), we get

$$g(\nabla_X \phi Y, Z) - g(\phi\nabla_X Y, Z) = g(\sigma(X, Y), \phi Z).$$

Therefore result follows from the above equation. □

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