

## A REMARK ON THE LORENTZIAN ALMOST CONTACT METRIC MANIFOLDS

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**ABSTRACT.** The object of the present paper is to introduce a new concept on the Lorentzian almost contact metric manifold and its some geometric properties have been studied. Also, an example has been constructed to support the Lorentzian almost contact metric manifolds.

### 1. INTRODUCTION

In 1989, Matsumoto [1] introduced the notion of LP-Sasakian manifold. Then Mihai and Rosca [2] introduced the same notion independently and they obtained several results in this manifold. LP-Sasakian manifolds have also been studied by Matsumoto and Mihai [3] and Yildiz, De and Ata [4].

Let  $M$  be an  $(2n + 1)$ -dimensional differentiable manifold endowed with a  $(1, 1)$  tensor field  $\phi$ , a vector field  $\xi$ , a 1-form  $\eta$  which satisfies

$$\phi^2(X) = X + \eta(X)\xi, \eta(\xi) = -1,$$

for all vector fields  $X, Y$ . Then such a structure  $(\phi, \eta, \xi)$  is termed as almost paracontact structure. A Lorentzian metric  $g$  of type  $(0, 2)$  such that for each point  $p \in M$ , the tensor  $g_p : T_p M \times T_p M \rightarrow \mathbb{R}$  is a non-degenerate inner product of signature  $(-, +, +, \dots, +)$ , where  $T_p M$  denotes the tangent space of  $M$  at  $p$  and  $\mathbb{R}$  is the real number space which satisfies

$$g(X, \xi) = \eta(X), g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y).$$

The manifold  $M$  with the structure  $(\phi, \eta, \xi, g)$  is called Lorentzian almost paracontact manifold [1].

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*Keywords:* LP-Sasakian manifold; Lorentzian almost contact metric manifold; generalized  $\eta$ -Einstein manifold;  $\eta$ -Einstein manifold; eigen vector .

A Lorentzian almost paracontact manifold  $M$  equipped with the structure  $(\phi, \eta, \xi, g)$  is called an LP-Sasakian manifold [1] if

$$(\nabla_X \phi)Y = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X,$$

where  $\nabla$  be the Levi-Civita connection.

In this paper we study the Lorentzian almost contact metric manifolds. The paper is organized as follows: After introduction in section 2, from the definition by means of the tensor equations it is easily verified that the structure is Lorentzian almost contact metric structure and we construct an example to verify the Lorentzian almost contact manifolds. Some properties of the curvature tensor and the Ricci tensor has been studied in Section 3 and Section 4 respectively. Finally, we have discussed the Gauss equation on the Lorentzian almost contact metric structure.

## 2. DEFINITION AND EXAMPLE

Let  $M$  be a  $(2n + 1)$ -dimensional manifold and  $\phi$ ,  $\xi$  and  $\eta$  be a  $(1, 1)$  tensor field, a vector field and a 1-form on  $M$  respectively. If  $\phi$ ,  $\xi$  and  $\eta$  satisfy the conditions

$$(1) \quad \eta(\xi) = -1,$$

$$(2) \quad \phi^2(X) = -X - \eta(X)\xi,$$

for any vector field  $X$  on  $M$  and it seems customary to include also

$$(3) \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0, \quad \text{rank} \phi = 2n,$$

then  $M$  is said to have a special type of almost contact structure  $(\phi, \eta, \xi)$  and is called a special type of almost contact manifold.

A Lorentzian metric  $g$  be a type  $(0, 2)$  which satisfies

$$(4) \quad g(X, \xi) = \eta(X),$$

$$(5) \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y).$$

The manifold  $M$  with the structure  $(\phi, \eta, \xi, g)$  is called an Lorentzian almost contact metric manifold.

Combining (2) and (5), it implies that

$$(6) \quad g(\phi X, Y) = -g(X, \phi Y).$$

Therefore,  $\phi$  is a skew-symmetric. In an almost paracontact manifold,  $\phi$  is symmetric and in an almost contact manifold,  $\phi$  is skew symmetric.

An Lorentzian almost contact metric manifold  $M$  equipped with the structure  $(\phi, \eta, \xi, g)$  is called an LC-Sasakian manifold if  $\phi$ ,  $\xi$  and  $\eta$  satisfy the conditions

$$(7) \quad (\nabla_X \eta)(Y) = g(X, \phi Y),$$

$$(8) \quad (\nabla_X \phi)(Y) = -g(X, Y)\xi + \eta(Y)X,$$

$$(9) \quad \nabla_X \xi = -\phi X,$$

where  $X, Y$  on  $M$  and  $\nabla$  be the Levi-Civita connection.

**Example 2.1.** *In this section we construct an example to support the LC-Sasakian manifolds with the structure  $(\phi, \xi, \eta, g)$ .*

We consider the 5-dimensional manifold  $M = \{(x, y, z, u, v) \in R^5\}$ , where  $(x, y, z, u, v)$  are the standard coordinate in  $R^5$ .

We choose the vector fields

$$e_1 = 2\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}, \quad e_4 = 2\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial z}, \quad e_5 = \frac{\partial}{\partial v},$$

which are linearly independent at each point of  $M$ .

Let  $g$  be the Lorentzian metric defined by

$$g(e_i, e_j) = 0, \quad i \neq j, \quad i, j = 1, 2, 3, 4, 5$$

and

$$g(e_1, e_1) = g(e_2, e_2) = g(e_4, e_4) = g(e_5, e_5) = 1, \quad g(e_3, e_3) = -1.$$

Let  $\eta$  be the 1-form defined by

$$\eta(Z) = g(Z, e_3),$$

for any  $Z \in \chi(M)$ .

Let  $\phi$  be the  $(1, 1)$ -tensor field defined by

$$\phi e_1 = e_2, \quad \phi e_2 = -e_1, \quad \phi e_3 = 0, \quad \phi e_4 = e_5, \quad \phi e_5 = -e_4.$$

Using the linearity of  $\phi$  and  $g$ , we have

$$\eta(e_3) = -1,$$

$$\phi^2(Z) = -Z - \eta(Z)e_3$$

and

$$g(\phi Z, \phi U) = g(Z, U) + \eta(Z)\eta(U),$$

for any  $U, Z \in \chi(M)$ . Hence  $e_3 = \xi$  and  $M(\phi, \xi, \eta, g)$  is a Lorentzian almost contact manifold.

Then we have

$$\begin{aligned} [e_1, e_2] &= 2e_3, & [e_1, e_3] &= [e_1, e_4] = [e_1, e_5] = [e_2, e_3] = 0, \\ [e_4, e_5] &= 2e_3, & [e_2, e_4] &= [e_2, e_5] = [e_3, e_4] = [e_3, e_5] = 0. \end{aligned}$$

The Riemannian connection  $\nabla$  of the metric tensor  $g$  is given by Koszul's formula which is given by

$$(10) \quad \begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) - g(X, [Y, Z]) \\ &\quad + g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Taking  $e_3 = \xi$  and using Koszul's formula we get the following

$$\begin{aligned} \nabla_{e_1} e_1 &= 0, & \nabla_{e_1} e_2 &= -e_3, & \nabla_{e_1} e_3 &= -e_2, & \nabla_{e_1} e_4 &= 0, & \nabla_{e_1} e_5 &= 0, \\ \nabla_{e_2} e_1 &= e_3, & \nabla_{e_2} e_2 &= 0, & \nabla_{e_2} e_3 &= e_1, & \nabla_{e_2} e_4 &= 0, & \nabla_{e_2} e_5 &= 0, \\ \nabla_{e_3} e_1 &= e_2, & \nabla_{e_3} e_2 &= -e_1, & \nabla_{e_3} e_3 &= 0, & \nabla_{e_3} e_4 &= e_5, & \nabla_{e_3} e_5 &= -e_4, \\ \nabla_{e_4} e_1 &= 0, & \nabla_{e_4} e_2 &= 0, & \nabla_{e_4} e_3 &= -e_5, & \nabla_{e_4} e_4 &= 0, & \nabla_{e_4} e_5 &= -e_3, \\ \nabla_{e_5} e_1 &= 0, & \nabla_{e_5} e_2 &= 0, & \nabla_{e_5} e_3 &= e_4, & \nabla_{e_5} e_4 &= e_3, & \nabla_{e_5} e_5 &= 0. \end{aligned}$$

From the above calculations, the manifold under consideration satisfies  $\eta(\xi) = -1$  and  $\nabla_X \xi = -\phi X$ . Therefore, the manifold is an LC-Sasakian manifold with the structure  $(\phi, \xi, \eta, g)$ .

### 3. SOME PROPERTIES ON CURVATURE TENSOR

Analogous to the definition of the curvature tensor  $R$  of  $M$  with respect to the Levi-Civita connection  $\nabla$  is given by

$$(11) \quad R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

where  $X, Y, Z \in \chi(M)$  and  $\chi(M)$  is the set of all differentiable vector fields on  $M$ .

Using (8) and (9) in (11), we get

$$(12) \quad R(X, Y)\xi = \eta(X)Y - \eta(Y)X.$$

**Proposition 3.1.** *Under the same assumption as the equation 12,*

$$(13) \quad R(\xi, Y)\xi = -Y - \eta(Y)\xi,$$

$$(14) \quad \eta(R(X, Y)Z) = \eta(Y)g(X, Z) - \eta(X)g(Y, Z),$$

$$(15) \quad \tilde{R}(X, Y, Z, \xi) = \eta(Y)g(X, Z) - \eta(X)g(Y, Z),$$

where  $\tilde{R}(X, Y, Z, U) = g(R(X, Y)Z, U)$ .

Since the proof of Proposition 3.1 follows by a routine calculation, we shall omit it.

**Definition 3.1.** Let  $\Pi$  be a nondegenerate tangent plane to  $M$  at  $p$  of the tangent space  $T_p(M)$ . The number [5]

$$(16) \quad K(\Pi) = K(X, Y) = \frac{\tilde{R}(X, Y, X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2}$$

is independent of the choice of basis  $X, Y$  for  $\Pi$  and is called the sectional curvature  $K(\Pi)$  of  $\Pi$ .

from (15) and (16), we calculate that

$$K(X, \xi) = 1.$$

That means, the sectional curvature of any plane section containing  $\xi$  of the LC-Sasakian manifolds is constant and this sectional curvature is called as  $\xi$ -sectional curvature of an LC-Sasakian manifold.

**Theorem 3.1.** The  $\xi$ - sectional curvature on an LC-Sasakian manifold is constant.

#### 4. SOME PROPERTIES ON RICCI TENSOR

**Theorem 4.1.** Let  $M^{2n+1}$  be an LC-Sasakian manifold. If  $M^{2n+1}$  is  $\eta$ -Einstein manifold, then we get  $p \leq q$ .

**Definition 4.1.** A semi-Riemannian manifold  $M$  is said to be an  $\eta$ -Einstein manifold if the following condition

$$(17) \quad S(X, Y) = pg(X, Y) + q\eta(X)\eta(Y),$$

holds on  $M$ , where  $p, q$  are smooth functions and  $S$  be a Ricci tensor.

Proof: From the equation (12) yields,

$$(18) \quad S(X, \xi) = -2n\eta(X),$$

Putting  $Y = \xi$  in (17) and using (1), (4) and (18), we can write that

$$p = q - 2n.$$

That means,

$$p \leq q.$$

Hence the proof of theorem is completed.

**Theorem 4.2.** If  $M^{2n+1}$  be an LC-Sasakian manifold, then  $\xi$  is an eigen vector corresponding to the eigen value is  $-2n$ .

Proof: From the equation (18), we can easily proved the theorem.

**Theorem 4.3.** *If the vector field  $X$  is an orthogonal to  $\xi$ , then  $\xi$  is an eigen vector corresponding to the eigen value is 0.*

Proof: Let  $X$  is an orthogonal to  $\xi$ , that means  $\eta(X) = 0$  and with the help of the equation (18), we can write  $S(X, \xi) = 0$ , which implies that  $\xi$  is an eigen vector corresponding to the eigen value is 0. Hence the Theorem is proved.

**Definition 4.2.** *A Lorentzian Para-Sasakian manifold  $M$  is said to be a generalized  $\eta$ -Einstein manifold [4] if the following condition*

$$S(X, Y) = pg(X, Y) + q\eta(X)\eta(Y) + c\Omega(X, Y),$$

holds on  $M$ , where  $p, q, c$  are smooth functions and  $\Omega(X, Y) = g(\phi X, Y)$ . If  $c = 0$ , then the manifold reduces to an  $\eta$ -Einstein manifold.

**Lemma 4.1.** [4] *If the Ricci tensor  $S$  of type  $(0, 2)$  of an LP-Sasakian manifold is non-vanishing and satisfies the relation*

$$S(Y, Z)S(X, W) - S(X, Z)S(Y, W) = \rho[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ + g(\phi X, W)g(Y, Z),$$

where  $\rho$  is non-zero scalar, then the manifold is a generalized  $\eta$ -Einstein manifold.

**Theorem 4.4.** *If the Ricci tensor  $S$  of type  $(0, 2)$  on an LC-Sasakian manifold is non-vanishing and satisfies the relation*

$$S(Y, Z)S(X, W) - S(X, Z)S(Y, W) = \rho[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ + g(\phi X, W)g(Y, Z),$$

where  $\rho$  is non-zero scalar, then the manifold is an  $\eta$ -Einstein manifold.

*Proof.* Putting  $Y = Z = \xi$  in (19), we obtain that

$$S(\xi, \xi)S(X, W) - S(X, \xi)S(\xi, W) = \rho[g(\xi, \xi)g(X, W) - g(X, \xi)g(\xi, W)] \\ + g(\phi X, W)g(\xi, \xi).$$

By virtue of (20), we see that

$$(21) \quad 2nS(X, W) - 4n^2\eta(X)\eta(W) = \rho[-g(X, W) - \eta(X)\eta(W)] - g(\phi X, W).$$

Now, (21) can be written as

$$(22) \quad S(X, W) = -\frac{\rho}{2n}g(X, W) + \frac{(4n^2 - \rho)}{2n}\eta(X)\eta(W) - \frac{1}{2n}g(\phi X, W).$$

Interchanging  $X$  and  $W$  by  $W$  and  $X$  in (22), we can write

$$(23) \quad S(X, W) = -\frac{\rho}{2n}g(X, W) + \frac{(4n^2 - \rho)}{2n}\eta(X)\eta(W) - \frac{1}{2n}g(\phi W, X).$$

Since  $\phi$  is skew-symmetric and adding (22) and (23), we conclude that

$$(24) \quad S(X, W) = -\frac{\rho}{2n}g(X, W) + \frac{(4n^2 - \rho)}{2n}\eta(X)\eta(W).$$

where  $p = -\frac{\rho}{2n}$  and  $q = \frac{(4n^2 - \rho)}{2n}$ .

That means, the manifold is an  $\eta$ -Einstein manifold. This completes the proof.

**Definition 4.3.** *A semi-Riemannian manifold is said to be Ricci-semisymmetric if  $R(X, Y).S = 0$ .*

**Definition 4.4.** *A semi-Riemannian manifold  $M$  is said to be an Einstein manifold if the following condition  $S(X, Y) = dg(X, Y)$  holds on  $M$ , where  $d$  be a smooth function.*

**Theorem 4.5.** *An LC-Sasakian manifold is Ricci-semi-symmetric iff the manifold is Einstein manifold.*

*Proof.* The manifold under consideration is Ricci-semisymmetric, that is,

$$(R(X, Y).S)(U, V) = 0$$

Then we have

$$(25) \quad S(R(X, Y)U, V) + S(U, R(X, Y)V) = 0.$$

Putting  $V = \xi$  in (25) and using (12), (13) and (14), it follows that

$$(26) \quad \begin{aligned} 2n\eta(X)g(Y, U) - 2n\eta(Y)g(X, U) + \eta(X)S(U, Y) \\ - \eta(Y)S(U, X) = 0. \end{aligned}$$

Again putting  $Y = \xi$  in (26) and using (1), (4) and (18), we get

$$S(U, X) = -2ng(X, U).$$

This result shows that the manifold is an Einstein manifold, where  $d = -2n$ .

Conversly if the manifold is an Einstein manifold, then the manifold is Ricci-semisymmetric  $(R(X, Y).S)(U, V) = 0$ . Hence the proof of the theorem is completed.

**Theorem 4.6.** *Einstein's equation without cosmological constant with perfect fluid the scalar curvature of the LC-Sasakian manifold is  $r = 2\kappa\sigma - 4n$ .*

*Proof.* General relativity flows from Einstein's equation [5] given by

$$(27) \quad S(X, Y) - \frac{1}{2}rg(X, Y) + \lambda g(X, Y) = \kappa T(X, Y),$$

where  $S(X, Y)$  is the Ricci tensor of type  $(0, 2)$  of the spacetime,  $r$  is the scalar curvature,  $T(X, Y)$  is the energy-momentum tensor of type  $(0, 2)$ ,  $\lambda$  is the cosmological constant and  $\kappa$

is the gravitational constant.

Einstein's equation without cosmological constant is given by

$$(28) \quad S(X, Y) - \frac{1}{2}rg(X, Y) = \kappa T(X, Y).$$

The equation (27) and (28) of Einstein imply that "matter determines the geometry of spacetimes and conversely that the motion of matter is determined by the metric tensor of the space which is not fiat [6]".

The energy-momentum tensor is said to describe a Perfect fluid [5] if

$$(29) \quad T(X, Y) = (\sigma + \rho)\eta(X)\eta(Y) + \rho g(X, Y),$$

where  $\sigma$  is the energy density and  $\rho$  is the isotropic pressure of the fluid.

Combining (28) and (29), it implies that

$$(30) \quad S(X, Y) - \left(\frac{1}{2}r + \kappa\rho\right)g(X, Y) = \kappa(\sigma + \rho)\eta(X)\eta(Y).$$

Putting  $Y = \xi$  in (30) and using (1, (4) and (18), it follows that

$$\left(\frac{1}{2}r - \kappa\sigma + 2n\right)\eta(X) = 0$$

That means,  $\eta(X) \neq 0$  and  $r = 2\kappa\sigma - 4n$ . This completes the proof of the theorem.

## 5. GAUSS FORMULA AND EQUATION

Let  $\bar{M}$  be an  $n$ -dimensional manifold isometrically immersed in an  $2n + 1$ -dimensional manifold  $M$ . We have the basic formula for submanifold [7]

$$(31) \quad \nabla_X Y = \bar{\nabla}_X Y + B(X, Y),$$

where  $\bar{\nabla}_X Y$  and  $B(X, Y)$  are the tangential component and the normal component of  $\nabla_X Y$  respectively. Then  $\bar{\nabla}$  is the operator of covariant differentiation with respect to the induced metric on  $\bar{M}$ . The formula (31) is called the Gauss formula.

**Theorem 5.1.** *If LC-Sasakian manifold is minimal, then the tangential component of the submanifold  $\bar{\nabla}_{e_i} e_i$  for all  $i = 1, 2, 3, \dots, n$ ; is vanished, where  $\{e_1, e_2, e_3, \dots, e_n\}$  be an orthonormal basis.*

Proof: Putting  $X = Y = e_i; i = 1, 2, \dots, n$ , in (31), we have

$$(32) \quad \nabla_{e_i} e_i = \bar{\nabla}_{e_i} e_i + B(e_i, e_i).$$



The equation (32) will be

$$(33) \quad \nabla_{e_i} e_i = \bar{\nabla}_{e_i} e_i + n\mu,$$

where  $\{e_1, e_2, e_3, \dots, e_n\}$  be an orthonormal basis in  $T_X(\bar{M})$  and the mean curvature vector  $\mu$  of  $\bar{M}$  is defined to be  $\mu = \frac{\text{Trace}B}{n}$  and  $\text{Trace}B = \sum_i B(e_i, e_i)$ . If  $\mu = 0$ , then the mean curvature vector is minimal.

From the Example (2.1) and the equation (33), we implies that

$$\bar{\nabla}_{e_i} e_i = -n\mu$$

If  $\mu = 0$ , then  $\bar{\nabla}_{e_i} e_i = 0$ ; for all  $i = 1, 2, \dots, n$ .

Thus the proof is completed.

The equation of Gauss [7] is

$$(34) \quad \begin{aligned} g(R(X, Y)Z, W) &= g(\bar{R}(X, Y)Z, W) - g(B(X, W), B(Y, Z)) \\ &\quad + g(B(Y, W), B(X, Z)), \end{aligned}$$

where  $\bar{R}$  and  $R$  are the curvature tensor fields of  $\bar{M}$  and  $M$  respectively.

**Theorem 5.2.** *The Gauss equation on the LC-Sasakian manifold is  $g(\bar{R}(X, Y)Z, W) = g(Y, U)g(X, Z) - g(X, U)g(Y, Z) + g(B(X, W), B(Y, Z)) - g(B(Y, W), B(X, Z))$ .*

Proof: From (15) and (34), we concluded that

$$\begin{aligned} g(\bar{R}(X, Y)Z, W) &= g(Y, U)g(X, Z) - g(X, U)g(Y, Z) + g(B(X, W), B(Y, Z)) \\ &\quad - g(B(Y, W), B(X, Z)). \end{aligned}$$

Hence the theorem is proved.

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