A COUPLED FIXED POINT THEOREM IN COMPLEX VALUED METRIC SPACES

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Abstract. In this paper, we introduce a common coupled fixed point theorem for weakly compatible mappings satisfying generalized contraction condition under rational expressions in complex valued metric spaces. Our theorem improve and extend some of the known results in the literature (see [11]). Finally, we give an example to support our theorem.

1. Introduction and Preliminaries

Fixed point theory is very useful theory in various branches as determining the existence and uniqueness of solutions to many mathematical equations in mathematical science, engineering and applications in other fields. Banach's contraction principle play an important role as the most widely used fixed point theorem in all analysis. In 2011, Azam et al. [1] introduced the concept of complex valued metric spaces which is more general than ordinary metric spaces. Recently, Both of Bhaskar and Laxikantham [3] introduced the concept of coupled fixed point. Also, some authors deduced coupled and common coupled fixed point theorems for two self-mappings in complex valued metric spaces for example [4,7,9,10,13].

In this paper, using the concept of $(E.A.)$ we establish a common fixed point theorem for weakly compatible mappings in the frame work of complex valued metric spaces. We recall some basic definitions and results that we need in our discussion.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } \operatorname{Re}(z_1) \leq \operatorname{Re}(z_2) \text{ and } \operatorname{Im}(z_1) \leq \operatorname{Im}(z_2).$$

Consequently, one can deduce that $z_1 \precsim z_2$ if one of the following conditions satisfied:

(p_1) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$,
(p_2) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(j_2)$,

(p_3) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,

(p_4) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

In particular, we write $z_1 \precsim z_2$ if $z_1 \neq z_2$ and one of (p_1), (p_2) and (p_3) is satisfied and we write $z_1 \prec z_2$ if only (p_3) is satisfied.

Remark 1.1: We can easily check the following notes:

(1) $a, b \in \mathbb{R}$, $a \leq b \Rightarrow az \precsim bz$ $\forall\, z \in \mathbb{C}$.

(2) $0 \precsim z_1 \precsim z_2 \Rightarrow |z_1| \precsim |z_2|$.

(3) $z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

Remark 1.2: Let (X, d) be a complex valued metric space. Then one can say

(1) $|d(x, y)|$ or $|d(u, v)| < |1 + d(x, y) + d(u, v)|$ $\forall\, x, y, u, v \in X$.

(2) $|d(x, y)| > 0$, if $x \neq y$.

Definition 1.1 [1] Let X be a nonempty set and $\mathbb{C}$ be the set of all complex numbers. A function $d : X \times X \to \mathbb{C}$ is called a complex valued metric on X if for all $x, y, z \in X$, the following conditions are satisfied:

(CVM_1) $0 \precsim d(x, y)$ and $d(x, y) = 0$ iff $x = y$,

(CVM_2) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(CVM_3) $d(x, y) \precsim d(x, z) + d(z, y)$ $\forall\, x, y, z \in X$.

Then (X, d) is called a complex valued metric space.

Example 1.1 [14] Let $X = \mathbb{C}$ be a set of complex numbers. Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(x_1, x_2) = e^{ik} |x_1 - x_2| \quad \forall\, x_1, x_2 \in X,$$

where $k \in [0, \frac{\pi}{2}]$. Then (X, d) is called a complex valued metric space.

Definition 1.2 [8] Let (X, d) be a complex valued metric space. The pair $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $S : X \times X \longrightarrow X$ if

$$x = S(x, y) \quad \text{and} \quad y = S(y, x).$$

Example 1.2 [11] Let $X = \mathbb{R}$ with a complex valued metric d defined as $d(x, y) = i |x - y|$. Let the mapping $S : X \times X \to X$ defined as $S(x, y) = x^2 y^3$. Then the pairs $(0, 0)$ and $(1, 1)$ are two coupled fixed points of S.

Definition 1.3 [6] Let $S : X \times X \longrightarrow CB(X)$ and $T : X \longrightarrow X$ be two given mappings. Then an element $(x, y) \in X \times X$ is called

(1) A coupled coincidence point of a pair (S, T), if $Tx \in S(x, y)$ and $Ty \in S(y, x)$,

(2) A common coupled fixed point of a pair (S, T), if

$$x = Tx \in S(x, y) \quad \text{and} \quad y = Ty \in S(y, x).$$

Definition 1.4 [2] Let X be a nonempty set and $F : X \times X \longrightarrow X$ and $S : X \longrightarrow X$. Then the pair (F, S) is called weakly compatible if $S(F(x, y)) = F(Sx, Sy)$ and $S(F(y, x)) = F(Sy, Sx)$, whenever $Sx = F(x, y)$ and $Sy = F(y, x)$.

Definition 1.5 [12] Let X be a nonempty set and $F : X \times X \longrightarrow X$ and $G : X \longrightarrow X$ be two self-mappings. Then the pair (F, G) is said to satisfy property $(E.A.)$ if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that for some $l, l' \in X$, we have

$$\lim_{n \to \infty} F(x_{2n}, y_{2n}) = \lim_{n \to \infty} Gx_{2n} = l, \quad \lim_{n \to \infty} F(y_{2n}, x_{2n}) = \lim_{n \to \infty} Gy_{2n} = l'.$$

Definition 1.6 [1] Let $\{x_n\}$ be a sequence in a complex valued metric space (X, d) and $x \in X$. Then,

(i) x is called the limit of $\{x_n\}$ if for every $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $d(x_n, x) \prec \varepsilon$ for all $n > n_0$ and we can write $\lim_{n \to \infty} x_n = x$.

(ii) $\{x_n\}$ is called a Cauchy sequence if for every $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+m}) \prec \varepsilon$ for all $n > n_0$, where $m \in \mathbb{N}$.

(iii) (X, d) is said to be a complete complex valued metric space if every Cauchy sequence is convergent in (X, d).

Lemma 1.1 [1] Let (X, d) be a complex valued metric space. Then a sequence $\{x_n\}$ in X converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma 1.2 [1] Let (X, d) be a complex valued metric space. Then a sequence $\{x_n\}$ in X is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.

Lemma 1.3 [5] Let (X, d) be a complex valued metric space and $\{x_n\}$ be a sequence such that $\lim_{n \to \infty} x_n = x$. Then $\lim_{n \to \infty} d(x_n, a) = d(x, a) \quad \forall \ a \in X$.

2. Main Results

In this section, we present a common coupled fixed point theorem in complex valued metric space:

Theorem 2.1 Let (X, d) be a complete complex valued metric space. Suppose $S, T : X \times X \to X$ and $P, Q : X \to X$ be self-mappings satisfy:

(i) $\quad d\big(S(x,y), T(u,v)\big) \precsim k_1 \dfrac{d(Px, Qu) + d(Py, Qv)}{2} + k_2 \dfrac{d(Px, S(x,y))\, d(Qu, T(u,v))}{1 + d(Px, Qu) + d(Py, Qv)}$

$\qquad + k_3 \dfrac{d(Qu, S(x,y))\, d(Px, T(u,v))}{1 + d(Px, Qu) + d(Py, Qv)},$ $\hspace{3cm}$ (1)

for all $x, y, u, v \in X$, where k_1, k_2, k_3 are non-negative reals with $0 \le k_1 + k_2 < 1$ and $0 \le k_1 + k_3 < 1$.

(ii) The pairs (S, P) and (T, Q) are weakly compatible and satisfy property $(E.A.)$.

(iii) $T(X \times X) \subseteq P(X)$ and $S(X \times X) \subseteq Q(X)$ where both of $P(X)$ and $Q(X)$ are closed sub-space of X.

Then S, T, P and Q have a unique common coupled fixed point in $X \times X$.

Proof. Let x_0, y_0 be arbitrary points in X . Since $T(X \times X) \subseteq P(X)$ and $S(X \times X) \subseteq Q(X)$, then we can define two sequences $\{x_n\}$ and $\{y_n\}$ in X such that,

$$
\begin{cases}
x_{2n+1} = Qx_{2n+1} = S(x_{2n}, y_{2n}), & x_{2n+2} = Px_{2n+2} = T(x_{2n+1}, y_{2n+1}) \\
y_{2n+1} = Qy_{2n+1} = S(y_{2n}, x_{2n}), & y_{2n+2} = Py_{2n+2} = T(y_{2n+1}, x_{2n+1})
\end{cases},
$$
(2)

Also, since the pairs (S, P) and (T, Q) satisfy property $(E.A.)$, then

$$
\begin{cases}
\lim\limits_{n\to\infty} S(x_{2n}, y_{2n}) = \lim\limits_{n\to\infty} Px_{2n} = l, & \lim\limits_{n\to\infty} S(y_{2n}, x_{2n}) = \lim\limits_{n\to\infty} Py_{2n} = l' \\
\lim\limits_{n\to\infty} T(x_{2n}, y_{2n}) = \lim\limits_{n\to\infty} Qx_{2n} = l, & \lim\limits_{n\to\infty} T(y_{2n}, x_{2n}) = \lim\limits_{n\to\infty} Qy_{2n} = l'
\end{cases},
$$
(3)

Hence, we have

$d(x_{2n+1}, x_{2n+2}) = d\big(S(x_{2n}, y_{2n}), T(x_{2n+1}, y_{2n+1})\big)$

$\precsim k_1 \dfrac{d(Px_{2n}, Qx_{2n+1}) + d(Py_{2n}, Qy_{2n+1})}{2}$

$\quad + k_2 \dfrac{d(Px_{2n}, S(x_{2n}, y_{2n}))\, d(Qx_{2n+1}, T(x_{2n+1}, y_{2n+1}))}{1 + d(Px_{2n}, Qx_{2n+1}) + d(Py_{2n}, Qy_{2n+1})}$

$\quad + k_3 \dfrac{d(Qx_{2n+1}, S(x_{2n}, y_{2n}))\, d(Px_{2n}, T(x_{2n+1}, y_{2n+1}))}{1 + d(Px_{2n}, Qx_{2n+1}) + d(Py_{2n}, Qy_{2n+1})}$

$$= k_1 \frac{d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1})}{2} + k_2 \frac{d(x_{2n}, x_{2n+1}) \, d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1})}$$

$$+ \, k_3 \frac{d(x_{2n+1}, x_{2n+1}) \, d(x_{2n}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1})}$$

$$= k_1 \frac{d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1})}{2} + k_2 \, d(x_{2n+1}, x_{2n+2}) \left[\frac{d(x_{2n}, x_{2n+1})}{1 + d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1})} \right]$$

$$\precsim k_1 \frac{d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1})}{2} + k_2 \, d(x_{2n+1}, x_{2n+2}).$$

This implies that

$$(1 - k_2) \, d(x_{2n+1}, x_{2n+2}) \precsim k_1 \frac{d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1})}{2},$$

that is,

$$d(x_{2n+1}, x_{2n+2}) \precsim \frac{k_1 \left[d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1}) \right]}{2(1 - k_2)}. \tag{4}$$

By a similar way, we obtain

$$d(y_{2n+1}, y_{2n+2}) \precsim \frac{k_1 \left[d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1}) \right]}{2(1 - k_2)}. \tag{5}$$

Adding (4) and (5), we have

$$d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}) \precsim \rho \left[d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1}) \right],$$

where $0 \le \rho = \dfrac{k_1}{1 - k_2} < 1$ as $0 \le k_1 + k_2 < 1$.

Similarly,

$$d(x_{2n+2}, x_{2n+3}) = d\big(T(x_{2n+1}, y_{2n+1}), S(x_{2n+2}, y_{2n+2})\big)$$

$$\precsim k_1 \frac{d(Px_{2n+2}, Qx_{2n+1}) + d(Py_{2n+2}, Qy_{2n+1})}{2}$$

$$+ \, k_2 \frac{d(Px_{2n+2}, S(x_{2n+2}, y_{2n+2})) \, d(Qx_{2n+1}, T(x_{2n+1}, y_{2n+1}))}{1 + d(Px_{2n+2}, Qx_{2n+1}) + d(Py_{2n+2}, Qy_{2n+1})}$$

$$+ \, k_3 \frac{d(Qx_{2n+1}, S(x_{2n+2}, y_{2n+2})) \, d(Px_{2n+2}, T(x_{2n+1}, y_{2n+1}))}{1 + d(Px_{2n+2}, Qx_{2n+1}) + d(Py_{2n+2}, Qy_{2n+1})}$$

$$= k_1 \frac{d(x_{2n+2}, x_{2n+1}) + d(y_{2n+2}, y_{2n+1})}{2} + k_2 \frac{d(x_{2n+2}, x_{2n+3}) \, d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n+2}, x_{2n+1}) + d(y_{2n+2}, y_{2n+1})}$$

$$+ \, k_3 \frac{d(x_{2n+1}, x_{2n+3}) \, d(x_{2n+2}, x_{2n+2})}{1 + d(x_{2n+2}, x_{2n+1}) + d(y_{2n+2}, y_{2n+1})}$$

$$= k_1 \frac{d(x_{2n+2}, x_{2n+1}) + d(y_{2n+2}, y_{2n+1})}{2}$$

$$+ k_2 \, d(x_{2n+2}, x_{2n+3}) \left[\frac{d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n+2}, x_{2n+1}) + d(y_{2n+2}, y_{2n+1})} \right]$$

$$\precsim k_1 \frac{d(x_{2n+2}, x_{2n+1}) + d(y_{2n+2}, y_{2n+1})}{2} + k_2 \, d(x_{2n+2}, x_{2n+3}).$$

Consequently, we have

$$d(x_{2n+2}, x_{2n+3}) \precsim \frac{k_1 \left[d(x_{2n+2}, x_{2n+1}) + d(y_{2n+2}, y_{2n+1}) \right]}{2(1 - k_2)}. \qquad (6)$$

Also, we can prove that

$$d(y_{2n+2}, y_{2n+3}) \precsim \frac{k_1 \left[d(x_{2n+2}, x_{2n+1}) + d(y_{2n+2}, y_{2n+1}) \right]}{2(1 - k_2)}. \qquad (7)$$

Adding (6) and (7), we have

$$d(x_{2n+2}, x_{2n+3}) + d(y_{2n+2}, y_{2n+3}) \precsim \rho \left[d(x_{2n+2}, x_{2n+1}) + d(y_{2n+2}, y_{2n+1}) \right].$$

Now, one can say that

$$d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2}) \precsim \rho \left[d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \right]$$
$$\precsim \rho^2 \left[d(x_{n-1}, x_n) + d(y_{n-1}, y_n) \right]$$
$$\precsim \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$
$$\precsim \rho^{n+1} \left[d(x_0, x_1) + d(y_0, y_1) \right].$$

Consequently, for any $m < n$, we have

$$d(x_m, x_n) + d(y_m, y_n) \precsim \left[d(x_m, x_{m+1}) + d(y_m, y_{m+1}) \right] + \left[d(x_{m+1}, x_n) + d(y_{m+1}, y_n) \right]$$
$$\precsim \left[d(x_m, x_{m+1}) + d(y_m, y_{m+1}) \right] + \left[d(x_{m+1}, x_{m+2}) + d(y_{m+1}, y_{m+2}) \right]$$
$$+ \left[d(x_{m+2}, x_n) + d(y_{m+2}, y_n) \right]$$
$$\precsim \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$
$$\precsim \left[d(x_m, x_{m+1}) + d(y_m, y_{m+1}) \right] + \left[d(x_{m+1}, x_{m+2}) + d(y_{m+1}, y_{m+2}) \right] +$$
$$\ldots \ldots \ldots \ldots + \left[d(x_{n-1}, x_n) + d(y_{n-1}, y_n) \right]$$
$$\precsim \left[\rho^m + \rho^{m+1} + \ldots \ldots \ldots + \rho^{n-1} \right] \left[d(x_0, x_1) + d(y_0, y_1) \right]$$
$$\precsim \rho^m \left[1 + \rho + \rho^2 + \ldots \ldots \ldots + \rho^{n-m-1} \right] \left[d(x_0, x_1) + d(y_0, y_1) \right]$$
$$\precsim \frac{\rho^m}{1 - \rho} \left[d(x_0, x_1) + d(y_0, y_1) \right] \longrightarrow 0, \text{ as } m \longrightarrow \infty.$$

i.e., $d(x_m, x_n) \longrightarrow 0$ and $d(y_m, y_n) \longrightarrow 0$ as $m, n \longrightarrow \infty$.

From that, we can deduce that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence in X.
Since (X, d) is complete, then there exists $x, y \in X$ such that $x_n \longrightarrow x$ and
$y_n \longrightarrow y$ as $n \longrightarrow \infty$. Then from (3), one can write

$$d\big(S(x, y), x\big) \precsim d\big(S(x, y), x_{2n+2}\big) + d\big(x_{2n+2}, x\big)$$
$$= d\big(S(x, y), T(x_{2n+1}, y_{2n+1})\big) + d\big(x_{2n+2}, x\big)$$

$$\precsim k_1 \frac{d(Px, Qx_{2n+1}) + d(Py, Qy_{2n+1})}{2}$$
$$+ k_2 \frac{d(Px, S(x,y))\, d(Qx_{2n+1}, T(x_{2n+1}, y_{2n+1}))}{1 + d(Px, Qx_{2n+1}) + d(Py, Qy_{2n+1})}$$
$$+ k_3 \frac{d(Qx_{2n+1}, S(x,y))\, d(Px, T(x_{2n+1}, y_{2n+1}))}{1 + d(Px, Qx_{2n+1}) + d(Py, Qy_{2n+1})} + d(x_{2n+2}, x)$$

$$\precsim k_1 \frac{d(x, x_{2n+1}) + d(y, y_{2n+1})}{2} + k_2 \frac{d(x, S(x,y))\, d(x_{2n+1}, x_{2n+2})}{1 + d(x, x_{2n+1}) + d(y, y_{2n+1})}$$
$$+ k_3 \frac{d(x_{2n+1}, S(x,y))\, d(x, x_{2n+2})}{1 + d(x, x_{2n+1}) + d(y, y_{2n+1})} + d(x_{2n+2}, x).$$

This implies that

$$d\big(S(x,y), x\big) \precsim 0, \quad \text{as} \quad n \longrightarrow \infty.$$

Then, $d\big(S(x,y), x\big) = 0$. i.e., $S(x,y) = x$.

By a similar way, we can prove that $S(y,x) = y$.

Also,
$$d\big(x, T(x,y)\big) = d\big(S(x,y), T(x,y)\big)$$
$$\precsim k_1 \frac{d(Px, Qx) + d(Py, Qy)}{2} + k_2 \frac{d\big(Px, S(x,y)\big)\, d\big(Qx, T(x,y)\big)}{1 + d(Px, Qx) + d(Py, Qy)}$$
$$+ k_3 \frac{d\big(Qx, S(x,y)\big)\, d\big(Px, T(x,y)\big)}{1 + d(Px, Qx) + d(Py, Qy)}$$
$$\precsim k_1 \frac{d(x,x) + d(y,y)}{2} + k_2 \frac{d(x,x)\, d\big(x, T(x,y)\big)}{1 + d(x,x) + d(y,y)} + k_3 \frac{d(x,x)\, d\big(x, T(x,y)\big)}{1 + d(x,x) + d(y,y)}.$$

This Lead us to $d\big(x, T(x,y)\big) \precsim 0$. Then, $d\big(x, T(x,y)\big) = 0$. i.e., $T(x,y) = x$.
By a similar way, we can deduce that $T(y,x) = y$.
Hence (x,y) is a common coupled fixed point of S and T.
To show the uniqueness, Suppose that $(x^*, y^*) \neq (x,y)$ be another common coupled fixed point of S and T.
i.e., $S(x^*, y^*) = T(x^*, y^*) = x^*$ and $S(y^*, x^*) = T(y^*, x^*) = y^*$.
Then from (1), we get
$$d(x, x^*) = d\big(S(x,y), T(x^*, y^*)\big)$$
$$\precsim k_1 \frac{d(Px, Qx^*) + d(Py, Qy^*)}{2} + k_2 \frac{d\big(Px, S(x,y)\big)\, d\big(Qx^*, T(x^*, y^*)\big)}{1 + d(Px, Qx^*) + d(Py, Qy^*)}$$
$$+ k_3 \frac{d\big(Qx^*, S(x,y)\big)\, d\big(Px, T(x^*, y^*)\big)}{1 + d(Px, Qx^*) + d(Py, Qy^*)}$$

$$\precsim k_1 \frac{d(x,x^*)+d(y,y^*)}{2} + k_2 \frac{d(x,x)\,d(x^*,x^*)}{1+d(x,x^*)+d(y,y^*)} + k_3 \frac{d(x^*,x)\,d(x,x^*)}{1+d(x,x^*)+d(y,y^*)}$$

$$\precsim k_1 \frac{d(x,x^*)+d(y,y^*)}{2} + k_3\, d(x,x^*)\left[\frac{d(x,x^*)}{1+d(x,x^*)+d(y,y^*)}\right].$$

This means that

$$d(x,x^*) \precsim \tfrac{k_1}{2}\left[d(x,x^*)+d(y,y^*)\right] + k_3\, d(x,x^*). \qquad (8)$$

Similarly, one can show that

$$d(y,y^*) \precsim \tfrac{k_1}{2}\left[d(x,x^*)+d(y,y^*)\right] + k_3\, d(y,y^*). \qquad (9)$$

Adding (8) and (9), we get

$$d(x,x^*)+d(y,y^*) \precsim (k_1+k_3)\left[d(x,x^*)+d(y,y^*)\right].$$

This implies that

$$(1-k_1-k_3)\left[d(x,x^*)+d(y,y^*)\right] \precsim 0.$$

Since $0 \le k_1 + k_3 < 1$ (i.e., $1 - k_1 - k_3 \ne 0$). Consequently,

$$\left[d(x,x^*)+d(y,y^*)\right] \precsim 0,$$

then $\left[d(x,x^*)+d(y,y^*)\right] = 0$. From that, we get

$$d(x,x^*) = 0 = d(y,y^*).$$

Thus $x = x^*$ and $y = y^*$. i.e., $(x,y) = (x^*,y^*)$ which is a contradiction. Hence (x,y) is a unique common coupled fixed point of S and T.

Now, if we take $P = Q = I$, (where I is the identity mapping) in the above theorem, we get the following corollary.

Corollary 2.1 [11] Let (X,d) be a complex valued metric space. Let $S,T : X \times X \to X$ such that:

$$d\big(S(x,y),T(u,v)\big) \precsim k_1 \frac{d(x,u)+d(y,v)}{2} + k_2 \frac{d(x,S(x,y))\,d(u,T(u,v))}{1+d(x,u)+d(y,v)}$$

$$+ k_3 \frac{d(u,S(x,y))\,d(x,T(u,v))}{1+d(x,u)+d(y,v)}, \qquad \forall\, x,y,u,v \in X,$$

where k_1, k_2, k_3 are non-negative reals satisfying $k_1 + k_2 < 1$ and $k_1 + k_3 < 1$. Then the mappings S and T have a unique common coupled fixed point in $X \times X$.

Example 2.1. Let (X, d) be a complete complex valued metric space, where $X = [0, 1]$ and define $d : X \times X \longrightarrow X$ by $d(x, y) = i\,|x - y|^2$. Define four mappings as follow:

(a) $S, T : X \times X \to X$ by $S(x, y) = \dfrac{x + y}{8}$ and $T(x, y) = \dfrac{x^2 + y^2}{16}$.

(b) $P, Q : X \to X$ by $Px = \dfrac{x}{2}$ and $Qx = \dfrac{x^2}{4}$

Previously, we find that

$$S(X \times X) = [0, \tfrac{1}{4}] \subseteq [0, \tfrac{1}{4}] = Q(X) \text{ and } T(X \times X) = [0, \tfrac{1}{8}] \subseteq [0, \tfrac{1}{2}] = P(X).$$

Then, both of $P(X)$ and $Q(X)$ are closed sub-space of X.

Let $x_n = \dfrac{1}{\sqrt{n}}$, $y_n = \dfrac{1}{\sqrt[3]{n}}$ be two sequences in X. Then

$$\lim_{n\to\infty} S(x_n, y_n) = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} S(y_n, x_n) = \lim_{n\to\infty} Py_n = 0,$$

and $\quad \lim_{n\to\infty} T(x_n, y_n) = \lim_{n\to\infty} Qx_n = \lim_{n\to\infty} T(y_n, x_n) = \lim_{n\to\infty} Qy_n = 0.$

Clearly, the pairs (S, P) and (T, Q) are weakly compatible and satisfy property $(E.A.)$.

By simple Calculation, we find

$$d(Px, Qu) = i\left|\frac{x}{2} - \frac{u^2}{4}\right|^2 = \frac{1}{16}\,i\,\left|2x - u^2\right|^2,$$

$$d(Py, Qv) = i\left|\frac{y}{2} - \frac{v^2}{4}\right|^2 = \frac{1}{16}\,i\,\left|2y - v^2\right|^2,$$

$$d\big(S(x, y), T(u, v)\big) = i\left|\frac{x + y}{8} - \frac{u^2 + v^2}{16}\right|^2 = \frac{1}{256}\,i\,\left|(2x - u^2) + (2y - v^2)\right|^2$$

$$\leq \frac{1}{256}\,i\{\left|2x - u^2\right|^2 + \left|2y - v^2\right|^2 + 2\left|2x - u^2\right|\left|2y - v^2\right|\}$$

$$\leq \frac{1}{256}\,i\{\left|2x - u^2\right|^2 + \left|2y - v^2\right|^2 + \left|2x - u^2\right|^2 + \left|2y - v^2\right|^2\}$$

$$\leq \frac{1}{128}\,i\{\left|2x - u^2\right|^2 + \left|2y - v^2\right|^2\} = \frac{1}{4}\,\frac{d(Px, Qu) + d(Py, Qv)}{2}.$$

Thus inequality (1) is satisfied with $k_1 = \tfrac{1}{4}$ and $k_2 = k_3 = 0$, for all $x, y, u, v \in X$. Also, $(0, 0)$ is the unique common coupled fixed point in $X \times X$.

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