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FIXED POINT THEOREMS FOR CONTRACTIVE MAPPINGS OF INTEGRAL TYPE

ROBAB HAMLBARANI HAGHI*, NEGAR BAKHSHI SADABADI

Department of Mathematics, Payame Noor University, P.O. Box 19395-3697, Tehran, Iran

*Email: robab.haghi@gmail.com

Abstract. In this paper, we define a class of functions and prove fixed point theorems under this class for contractive mappings of integral type in complete metric space. Examples are included. Finally, we discuss the application of our main results in the research of functional equations.

1. Introduction

Branciari [5] was the first to study the existence of fixed points for the contractive mappings of integral type. He established a nice integral version of the Banach contraction principle [3].
Afterwards, many authors continued the study of Branciari [5] and obtained many fixed point theorems for several classes of contractive mappings of integral type.see,e.g [6, 10, 13].
Throughout this paper, we assume that $\mathbf{R}^+ = [0, \infty)$ and
$\Phi_1 = \{\varphi : \varphi : \mathbf{R}^+ \to \mathbf{R}^+$is Lebesgue integrable, summable on each compact subset of $\mathbf{R}^+$ and $\int_0^\epsilon \phi(t)\mathbf{dt} > 0$ for each $\epsilon > 0\}$,
$\Phi_2 = \{\varphi : \varphi : \mathbf{R}^+ \to \mathbf{R}^+$satisfies that $\liminf_{n\to\infty} \varphi(a_n) > 0$ if and only if $\liminf_{n\to\infty}(a_n) > 0$ for each $\{a_n\} \subseteq \mathbf{R}^+\}$,
$\Phi_3 = \{\varphi : \varphi : \mathbf{R}^+ \to \mathbf{R}^+$ is nondecreasing continuous and $\varphi(t) = 0$ if and only if $t = 0\}$,
$\Phi_4 = \{\varphi : \varphi : \mathbf{R}^+ \to [0, 1)$ satisfy that $\limsup_{s\to t} \varphi(s) < 1$ for each $t > 0\}$,
The following lemmas play important roles in this paper.

Lemma 1.1. [9] *Let $\varphi \in \Phi_1$ and $\{r_n\}_{n\in\mathbf{N}}$ be a nonnegative sequence with $\lim_{n\to\infty} r_n = a$. Then*

$$\lim_{n\to\infty} \int_0^{r_n} \varphi(t)\mathbf{dt} = \int_0^a \varphi(t)\mathbf{dt}.$$

Keywords: contractive mappings of integral type, directed graph, functional equation.

Lemma 1.2. [9] *Let $\varphi \in \Phi_1$ and $\{r_n\}_{n\in\mathbf{N}}$ be a nonnegative sequence. Then*

$$\lim_{n\to\infty} \int_0^{r_n} \varphi(t)\mathbf{dt} = 0$$

if and only if $\lim\limits_{n\to\infty} r_n = 0$.

Lemma 1.3. [9] *Let $\varphi \in \Phi_2$. Then $\varphi(t) > 0$ if and only if $t > 0$.*

2. Main Results

In this section we show the existence and uniqueness of fixed points for contractive mappings of integral type. Now we give the following definition.

Definition 2.1. *Let ζ be the set of functions $g : (0,\infty)^3 \to (0,\infty)$ satisfying the following conditions:*

1. *g is continuous,*
2. *if $\int_0^v \varphi(t)\mathbf{d}t < \int_0^{g(u,u,v)} \varphi(t)\mathbf{d}t$, then $\int_0^v \varphi(t)\mathbf{d}t < \int_0^u \varphi(t)\mathbf{d}t$,*
3. *if $\int_0^u \varphi(t)\mathbf{d}t \le k\int_0^{g(u,0,0)} \varphi(t)\mathbf{d}t$ or $\int_0^u \varphi(t)\mathbf{d}t \le k\int_0^{g(0,0,u)} \varphi(t)\mathbf{d}t$ for all $k \in (0,1)$, then $u = 0$.*

As examples, the following functions belong to ζ

(1) If $g(a,b,c) = \max\{a,b,c\}$, then $g \in \zeta$.

Proof. obviously, g is continuous. If$\int_0^v \varphi(t)\mathbf{d}t < \int_0^{\max\{u,u,v\}} \varphi(t)\mathbf{d}t = \max\{\int_0^u \varphi(t)\mathbf{d}t, \int_0^v \varphi(t)\mathbf{d}t\}$, then we have $\int_0^v \varphi(t)\mathbf{d}t < \int_0^u \varphi(t)\mathbf{d}t$.
Also if

$$\int_0^u \varphi(t)\mathbf{d}t \le k\int_0^{\max\{0,0,u\}} \varphi(t)\mathbf{d}t = k\max\{\int_0^0 \varphi(t)\mathbf{d}t, \int_0^u \varphi(t)\mathbf{d}t\} = k\int_0^u \varphi(t)\mathbf{d}t,$$

so $(1-k)\int_0^u \varphi(t)\mathbf{d}t \le 0$, then $u = 0$. □

(2) $g(a,b,c) = \frac{a+b}{2}$,
(3) $g(a,b,c) = a$.

Theorem 2.2. *Let f be a mapping from a complete metric space (X,d) into itself satisfying*

$$\int_0^{d(fx,fy)} \varphi(t)\mathbf{dt} \le \alpha(M(x,y))\int_0^{M(x,y)} \varphi(t)\mathbf{dt}, \quad x,y \in X,$$

where $M(x,y) = g(d(x,y), d(x,fx), d(y,fy))$ and $\varphi \in \Phi_1$ and $\alpha : \mathbf{R}^+ \to [0,1)$ is a function with

$$\limsup_{s\to t} \alpha(s) < 1, \quad t > 0.$$

Then f has a unique fixed point.

Proof. Let $x \in X$, $n \in \mathbf{N}$ and $x_{n+1} = fx_n$ and let $d_n = d(fx_n, fx_{n+1})$. Since $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ we get $\int_0^{d_{n_0}} \varphi(t)\mathbf{dt} > 0$, for all $n_0 \in \mathbf{N}$. We show that $d_n \leq d_{n-1}$. On the contrary, suppose that there exists $n_0 \in \mathbf{N}$ such that $d_{n_0} > d_{n_0-1}$. We have

$$\int_0^{d_{n_0-1}} \varphi(t)\mathbf{dt} \leq \int_0^{d_{n_0}} \varphi(t)\mathbf{dt} = \int_0^{d(fx_{n_0}, fx_{n_0+1})} \varphi(t)\mathbf{dt}$$

$$\leq \alpha(M(x_{n_0}, x_{n_0+1})) \int_0^{M(x_{n_0}, x_{n_0+1})} \varphi(t)\mathbf{dt}, \tag{2.1}$$

where

$$M(x_{n_0}, x_{n_0+1}) = g(d(x_{n_0}, x_{n_0+1}), d(x_{n_0}, fx_{n_0}), d(x_{n_0+1}, fx_{n_0+1})),$$

from 2.1

$$\int_0^{d_{n_0-1}} \varphi(t)\mathbf{dt} \leq \int_0^{d_{n_0}} \varphi(t)\mathbf{dt} < \int_0^{g(d_{n_0-1}, d_{n_0-1}, d_{n_0})} \varphi(t)\mathbf{dt},$$

so from definition 2.1, we have

$$\int_0^{d_{n_0-1}} \varphi(t)\mathbf{dt} \leq \int_0^{d_{n_0}} \varphi(t)\mathbf{dt} < \int_0^{d_{n_0-1}} \varphi(t)\mathbf{dt},$$

which is impossible. So the sequence $\{d_n\}_{n\in\mathbf{N}}$ is the nonnegative non increasing which implies that there exists a constant $c \geq 0$ such that $\lim_{n\to\infty} d_n = c$. Suppose $c > 0$, then

$$\int_0^{d_n} \varphi(t)\mathbf{dt} = \int_0^{d(fx_n, fx_{n+1})} \varphi(t)\mathbf{dt} \leq \alpha(M(x_n, x_{n+1})) \int_0^{M(x_n, x_{n+1})} \varphi(t)\mathbf{dt}, \tag{2.2}$$

where $M(x_n, x_{n+1}) = g(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}))$. Then

$$\int_0^{d_n} \varphi(t)\mathbf{dt} < \int_0^{g(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}))} \varphi(t)\mathbf{dt} = \int_0^{g(d_{n-1}, d_{n-1}, d_n)} \varphi(t)\mathbf{dt},$$

so from definition 2.1, we have

$$\int_0^{d_n} \varphi(t)\mathbf{dt} < \int_0^{d_{n-1}} \varphi(t)\mathbf{dt},$$

taking $n \to \infty$

$$\int_0^{c} \varphi(t)\mathbf{dt} < \int_0^{c} \varphi(t)\mathbf{dt},$$

that is a contradiction. Then $c = 0$. Now, we show that $\{x_n\}$ is a cauchy sequence. Contrary, suppose that $\{x_n\}$ is not a cauchy sequence. Then there exists $\delta > 0$ such that for all $k > 0$ there exists $m(k) > n(k) > k$ with $d(x_{m_k}, x_{n_k}) \geq \delta$ and $d(x_{m_{k-1}}, x_{n_k}) < \delta$. We have

$$\begin{aligned} \delta \leq d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{m_{k-1}}) + d(x_{m_{k-1}}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{m_{k-1}}) + \delta, \end{aligned} \tag{2.3}$$

since $\lim_{k\to\infty} d(x_{m_k}, x_{m_k-1}) = 0$, by letting $k \to \infty$, we have $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \delta$. So

$$\begin{aligned}
\int_0^{\delta} \varphi(t)\mathbf{dt} &= \limsup_{k\to\infty} \int_0^{d(fx_{m_k}, fx_{n_k})} \varphi(t)\mathbf{dt} \\
&\leq \limsup_{k\to\infty} (\alpha(M(x_{m_k}, x_{n_k}))) \int_0^{M(x_{m_k}, x_{n_k})} \varphi(t)\mathbf{dt} \\
&= \limsup_{k\to\infty} (\alpha(M(x_{m_k}, x_{n_k}))) \int_0^{g(\delta,0,0)} \varphi(t)\mathbf{dt}, \qquad (2.4)
\end{aligned}$$

where $M(x_{m_k}, x_{n_k}) = g(d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{n_{k+1}}))$. From definition 2.1, $\delta = 0$. That is a contradiction. Hence $\{x_n\}$ is a cauchy sequence. Since (X, d) is a complete metric space, there exists $a \in X$ such that $x_n \to a$. We have

$$\int_0^{d(x_{n+1}, f(a))} \varphi(t)\mathbf{dt} \leq \alpha(M(x_n, a)) \int_0^{M(x_n,a)} \varphi(t)\mathbf{dt},$$

where

$$M(x_n, a) = g(d(x_n, a), d(x_n, f(x_n)), d(a, f(a))).$$

Letting $n \to \infty$

$$\int_0^{d(a,fa)} \varphi(t)\mathbf{dt} \leq \lim_{n\to\infty} \alpha(M(x_n, a)) \int_0^{g(0,0,d(a,fa))} \varphi(t)\mathbf{dt},$$

From definition 2.1, $d(a, fa) = 0$. Now, we prove uniqueness, suppose that f has another fixed point $b \in X$ with $a \neq b$. It follows that

$$\int_0^{d(a,b)} \varphi(t)\mathbf{dt} = \int_0^{d(fa,fb)} \varphi(t)\mathbf{dt} \leq \alpha(M(a, b)) \int_0^{M(a,b)} \varphi(t)\mathbf{dt}.$$

Where $M(a, b) = g(d(a, b), d(a, f(a)), d(b, f(b)))$. So

$$\int_0^{d(a,b)} \varphi(t)\mathbf{dt} \leq \alpha(g(d(a, b), 0, 0)) \int_0^{g(d(a,b),0,0)} \varphi(t)\mathbf{dt},$$

hence $d(a, b) = 0$. That is, $a = b$. $\square$

Example 2.3. *Let $X = [\frac{1}{2}, 1]$ be endowed with euclidean metric $d = |.|$. Assume that $f : X \to X$ and $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ and $\alpha : \mathbf{R}^+ \to [0, 1)$ are defined by $f(x) = \frac{x}{2}$ and $\varphi(t) = 2t$ and*

$$\alpha(t) = \begin{cases} \frac{1}{3} + \frac{t^2}{2}, & t \in [0, 1] \\ \frac{1}{2t}, & t \in (1, 3] \\ \frac{1}{t^{\frac{1}{2}}}, & t \in (3, \infty) \end{cases}$$

also put $g : (0, \infty)^3 \to (0, \infty)$ with $g(a, b, c) = \max\{a, b, c\}$. Clearly (X, d) is a complete metric space and $(\varphi, \alpha) \in \Phi_1 \times \Phi_4$. Let $x, y \in X$ and $x > y$. We have

$$M(x, y) = g(|x - y|, |x - fx|, |y - fy|) = g(|x - y|, |x - \frac{x}{2}|, |y - \frac{y}{2}|),$$

we have two cases

(1) *if* $M(x,y) = |x-y|$ *then*

$$\begin{aligned} \int_0^{d(fx,fy)} \varphi(t)\mathbf{dt} &= \int_0^{\frac{x-y}{2}} 2t\mathbf{dt} = \frac{|x-y|^2}{4} \\ &\leq (\frac{1}{3} + \frac{1}{2}|x-y|^2)|x-y|^2 \\ &= \alpha(M(x,y)) \int_0^{M(x,y)} \varphi(t)\mathbf{dt}. \end{aligned}$$

(2) *if* $M(x,y) = |\frac{x}{2}|$ *then*

$$\begin{aligned} \int_0^{d(fx,fy)} \varphi(t)\mathbf{dt} &= \int_0^{\frac{x-y}{2}} 2t\mathbf{dt} = \frac{|x-y|^2}{4} \\ &\leq (\frac{1}{3} + \frac{1}{8}|x|^2)\frac{|x|^2}{4} \\ &= \alpha(M(x,y)) \int_0^{M(x,y)} \varphi(t)\mathbf{dt}. \end{aligned}$$

So all the conditions of theorem 2.2 are satisfied. Then f has a unique fixed point in X. We see that $0 \in X$ is the unique fixed point of f.

In theorem 2.2 we chose $g(a,b,c) = a$, we can obtain the following theorem.

Theorem 2.4. [9] *Let f be a mapping from a complete metric space (X,d) into itself satisfying*

$$\int_0^{d(fx,fy)} \varphi(t)\mathbf{dt} \leq \alpha(d(x,y)) \int_0^{d(x,y)} \varphi(t)\mathbf{dt}, \quad x,y \in X,$$

where $\phi \in \Phi_1$ and $\alpha : \mathbf{R}^+ \to [0,1)$ is a function with

$$\limsup_{s \to t} \alpha(s) < 1, \quad t > 0.$$

Then f has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} f^n(x) = a$ for each $x \in X$.

Theorem 2.5. *Let f be mapping from a complete metric space (X,d) into itself satisfying*

$$\Psi(\int_0^{d(fx,fy)} \varphi(t)\mathbf{dt}) \leq \Psi(\int_0^{M(x,y)} \varphi(t)\mathbf{dt}) - \phi(\int_0^{M(x,y)} \varphi(t)\mathbf{dt}), \quad x,y \in X,$$

where $M(x,y) = g(d(x,y), d(x,fx), d(y,fy))$ and $\varphi \in \Phi_1$ and $\phi \in \Phi_2, \Psi \in \Phi_3$. Then f has a unique fixed point.

Proof. Let $x \in X$, $n \in \mathbf{N}$ and $x_{n+1} = fx_n$ and $d_n = d(fx_n, fx_{n+1})$. First we show that $d_n \leq d_{n-1}$, for all $n \in \mathbf{N}$. Suppose on the contrary that there exists $n_0 \in \mathbf{N}$ such that

$d_{n_0} > d_{n_0-1}$. Since $\varphi \in \Phi_1$, then $\int_0^{d_{n_0}} \varphi(t)\mathbf{dt} > 0$. We have

$$\begin{aligned}\Psi(\int_0^{d_{n_0-1}} \varphi(t)\mathbf{dt}) &< \Psi(\int_0^{d_{n_0}} \varphi(t)\mathbf{dt}) = \Psi(\int_0^{d(fx_{n_0},fx_{n_0+1})} \varphi(t)\mathbf{dt}) \\ &\leq \Psi(\int_0^{M(x_{n_0},x_{n_0+1})} \varphi(t)\mathbf{dt}) - \phi(\int_0^{M(x_{n_0},x_{n_0+1})} \varphi(t)\mathbf{dt}) \qquad (2.5)\end{aligned}$$

where $M(x_{n_0}, x_{n_0+1}) = g(d(x_{n_0}, x_{n_0+1}), d(x_{n_0}, x_{n_0+1}), d(x_{n_0+1}, x_{n_0+2}))$. since $\psi \in \Phi_3$, then

$$\int_0^{d_{n_0-1}} \varphi(t)\mathbf{dt} < \int_0^{d_{n_0}} \varphi(t)\mathbf{dt} < \int_0^{g(d_{n_0-1},d_{n_0-1},d_{n_0})} \varphi(t)\mathbf{dt},$$

from definition 2.1

$$\int_0^{d_{n_0-1}} \varphi(t)\mathbf{dt} < \int_0^{d_{n_0}} \varphi(t)\mathbf{dt} < \int_0^{d_{n_0-1}} \varphi(t)\mathbf{dt},$$

that is a contradiction. So $\{d_n\}$ is nonnegative and nonincreasing, which means that there exists a constant $c \geq 0$ such that $\lim_{n\to\infty} d_n = c$. Suppose $c > 0$. It follows that

$$\Psi(\int_0^{d_n} \varphi(t)\mathbf{dt}) = \Psi(\int_0^{d(fx_n,fx_{n+1})} \varphi(t)\mathbf{dt}) \leq \Psi(\int_0^{M(x_n,x_{n+1})} \varphi(t)\mathbf{dt}) - \phi(\int_0^{M(x_n,x_{n+1})} \varphi(t)\mathbf{dt}), \qquad (2.6)$$

where $M(x_n, x_{n+1}) = g(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}))$. since $\psi \in \Phi_3$, then

$$\int_0^{d_n} \varphi(t)\mathbf{dt} < \int_0^{g(d_{n-1},d_{n-1},d_n)} \varphi(t)\mathbf{dt},$$

from definition 2.1

$$\int_0^{d_n} \varphi(t)\mathbf{dt} < \int_0^{d_{n-1}} \varphi(t)\mathbf{dt},$$

letting $n \to \infty$, we have

$$\int_0^{c} \varphi(t)\mathbf{dt} < \int_0^{c} \varphi(t)\mathbf{dt},$$

which is impossible. Hence $c = 0$. Now we show that $\{x_n\}$ is a cauchy sequence. Suppose on the contrary that $\{x_n\}$ is not a cauchy sequence. So there exists $\delta > 0$ such that for all $k > 0$ there exists $m(k) > n(k) > k$ with $d(x_{m_k}, x_{n_k}) \geq \delta$ and $d(x_{m_{k-1}}, x_{n_k}) < \delta$. Similar to the theorem 2.2, we can get $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \delta$. Then

$$\Psi(\int_0^{d(fx_{m_k},fx_{n_k})} \varphi(t)\mathbf{dt}) \leq \Psi(\int_0^{M(x_{m_k},x_{n_k})} \varphi(t)\mathbf{dt}) - \phi(\int_0^{M(x_{m_k},x_{n_k})} \varphi(t)\mathbf{dt}),$$

where $M(x_{m_k}, x_{n_k}) = g(d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{n_k+1}))$. So

$$\Psi(\int_0^{d(fx_{m_k},fx_{n_k})} \varphi(t)\mathbf{dt}) < \Psi(\int_0^{g(d(x_{m_k},x_{n_k}),d(x_{m_k},x_{m_{k+1}}),d(x_{n_k},x_{n_k+1}))} \varphi(t)\mathbf{dt}),$$

since $\psi \in \Phi_3$

$$\int_0^{d(fx_{m_k},fx_{n_k})} \varphi(t)\mathbf{dt} < \int_0^{g(d(x_{m_k},x_{n_k}),d(x_{m_k},x_{m_{k+1}}),d(x_{n_k},x_{n_k+1}))} \varphi(t)\mathbf{dt},$$

since g is continuous and by letting $n \to \infty$

$$\int_0^{\delta} \varphi(t)\mathbf{dt} < \int_0^{g(\delta,0,0)} \varphi(t)\mathbf{dt}.$$

From definition 2.1, $\delta = 0$. That is a contradiction. Then $\{x_n\}$ is a cauchy sequence. Since (X, d) is a complete metric space, there exists $a \in X$, such that $x_n \to a$. We infer that

$$\Psi(\int_0^{d(x_{n+1},fa)} \varphi(t)\mathbf{dt}) \leq \Psi(\int_0^{M(x_n,a)} \varphi(t)\mathbf{dt}) - \phi(\int_0^{M(x_n,a)} \varphi(t)\mathbf{dt}), \tag{2.7}$$

Where $M(x_n, a) = g(d(x_n, a), d(x_n, x_{n+1}), d(a, fa))$. Then

$$\Psi(\int_0^{d(x_{n+1},fa)} \varphi(t)\mathbf{dt}) < \Psi(\int_0^{g(d(x_n,a),d(x_n,x_{n+1}),d(a,fa))} \varphi(t)\mathbf{dt}),$$

since ψ is nondecreasing

$$\int_0^{d(x_{n+1},fa)} \varphi(t)\mathbf{dt} < \int_0^{g(d(x_n,a),d(x_n,x_{n+1}),d(a,fa))} \varphi(t)\mathbf{dt},$$

letting $n \to \infty$

$$\int_0^{d(a,fa)} \varphi(t)\mathbf{dt} < \int_0^{g(0,0,d(a,fa))} \varphi(t)\mathbf{dt},$$

from definition 2.1, $d(a, fa) = 0$. Finally, we show the uniqueness. Suppose that f has another fixed point $b \in X$ with $a \neq b$. It follows that

$$\Psi(\int_0^{d(a,b)} \varphi(t)\mathbf{dt}) = \Psi(\int_0^{d(fa,fb)} \varphi(t)\mathbf{dt}) \leq \Psi(\int_0^{M(a,b)} \varphi(t)\mathbf{dt}) - \Phi(\int_0^{M(a,b)} \varphi(t)\mathbf{dt}), \tag{2.8}$$

where

$$M(a, b) = g(d(a, b), d(a, fa), d(b, fb)).$$

Hence from 2.8 we deduce that

$$\Psi(\int_0^{d(a,b)} \varphi(t)\mathbf{dt}) < \Psi(\int_0^{g(d(a,b),d(a,fa),d(b,fb))} \varphi(t)\mathbf{dt}),$$

then

$$\int_0^{d(a,b)} \varphi(t)\mathbf{dt} < \int_0^{g(d(a,b),0,0)} \varphi(t)\mathbf{dt},$$

hence $d(a, b) = 0$, and this completes the proof. □

Example 2.6. *Let $X = [\frac{1}{2}, 1]$ be endowed with euclidean metric $d = |.|$. Assume that $f : X \to X$ and $\varphi, \phi, \Psi : \mathbf{R}^+ \to \mathbf{R}^+$ are defined by $f(x) = \frac{x}{2}$,*

$$\varphi(t) = \begin{cases} \frac{t}{2}, & x \in [0, 1] \\ 1, & x \in (1, \infty) \end{cases} \qquad \phi(t) = \begin{cases} \frac{t^2}{4}, & x \in [0, 1] \\ \frac{t^2}{8}, & x \in (1, \infty) \end{cases}$$

and

$$\Psi(t) = \begin{cases} t, & x \in [0,1] \\ \frac{t^2+1}{2}, & x \in (1,\infty) \end{cases}$$

Put $g(a,b,c) = \max\{a,b,c\}$. *Clearly* (X,d) *is a complete metric space and* $(\varphi,\phi,\Psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$. *Let* $x, y \in X$ *and* $x > y$. *We have*

$$M(x,y) = g(|x-y|, |x-fx|, |y-fy|) = g(|x-y|, |\frac{x}{2}|, |\frac{y}{2}|),$$

we have two cases

(1) *If* $M(x,y) = |x-y|$, *then*

$$\begin{aligned} \Psi(\int_0^{d(fx,fy)} \varphi(t)\mathbf{dt}) &= \Psi(\int_0^{\frac{1}{2}|x-y|} \varphi(t)\mathbf{dt}) = \Psi(\frac{|x-y|^2}{16}) = \frac{|x-y|^2}{16} \\ &\leq \frac{|x-y|^2}{4} - \frac{|x-y|^4}{16} = \Psi(\frac{|x-y|^2}{4}) - \phi(\frac{|x-y|^2}{4}) \\ &= \Psi(\int_0^{|x-y|} \varphi(t)\mathbf{dt}) - \phi(\int_0^{|x-y|} \varphi(t)\mathbf{dt}) \\ &= \Psi(\int_0^{d(x,y)} \varphi(t)\mathbf{dt}) - \phi(\int_0^{d(x,y)} \varphi(t)\mathbf{dt}). \end{aligned}$$

(2) *If* $M(x,y) = |\frac{x}{2}|$, *then*

$$\begin{aligned} \Psi(\int_0^{d(fx,fy)} \varphi(t)\mathbf{dt}) &= \Psi(\int_0^{\frac{1}{2}|x-y|} \frac{t}{2}\mathbf{dt}) = \frac{|x-y|^2}{16} \leq \frac{|x|^2}{16} - \frac{|x|^4}{1024} \\ &= \Psi(\int_0^{|\frac{x}{2}|} \varphi(t)\mathbf{dt}) - \phi(\int_0^{|\frac{x}{2}|} \varphi(t)\mathbf{dt}) \end{aligned}$$

Thus by theorem 2.4, f *has a unique fixed point in* X. *We see that* 0 *is the fixed point of* f.

In theorem 2.5 we chose $g(a,b,c) = a$, we can obtain the following theorem.

Theorem 2.7. [9] *Let* f *be a mapping from a complete metric space* (X,d) *into itself satisfying*

$$\int_0^{d(fx,fy)} \varphi(t)\mathbf{dt} \leq \psi(\int_0^{d(x,y)} \varphi(t)\mathbf{dt}) - \phi(\int_0^{d(x,y)} \varphi(t)\mathbf{dt}), \quad x, y \in X,$$

where $\varphi \in \Phi_1$, $\phi \in \Phi_2$ *and* $\psi \in \Phi_3$. *Then* f *has a unique fixed point* $a \in X$ *such that* $\lim_{n\to\infty} f^n(x) = a$ *for each* $x \in X$.

3. Fixed point theorem endowed with a graph

In 2007, Jachymski [8] introduced the concept of G-contraction on a metric space endowed with graph G. Later on many authors undertook further investigations in this direction [1, 2, 4, 12].

Let (X,d) be a metric space and $\Delta = \{(x,x); \quad x \in X\}$. Consider a directed graph G with the set $v(G) = X$ and the set $E(G)$ of its edges contains all loops, that is $\delta \subseteq E(G)$. Assume that G has no parallel edges. Now we can identify G with the pair $(v(G), E(G))$. The graph G can be converted to a weighted graph by assigning to each edge a weight equal to the distance between its vertices.

Let G^{-1} denote conversion of the graph G obtained from graph G by reversing the direction of edges. Thus we have $v(G^{-1}) = v(G)$ and $E(G^{-1}) = \{(x, y) \in X \times X; \quad (y, x) \in E(G)\}$. The letter $\bar{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. It is convenient to treat $\bar{G}$ as a directed graph for which the set of its edges is symmetric. That is $E(\bar{G}) = E(G) \cup E(G^{-1})$.

If x, y are vertices in a graph G, then a path in G from x to y of length l is a sequence $\{x_i\}_{i=0}^{l}$ of $l+1$ vertices such that $x_0 = x$, $x_l = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, ..., l$.

A graph G is called connected if there is a path between any two vertices of G. G is weakly connected if $\bar{G}$ is connected. The space (X, d) is said to have property P if Whenever a sequence $\{x_n\}$ in X, convergence to x and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbf{N}$, then there exists a subsequence $\{x_{n_k}\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbf{N}$. G is transitive if for any two vertices x and y that are connected by a directed finite path, we have $(x, y) \in E(G)$. Suppose that $Fixf$ be all fixed points of f.

Theorem 3.1. *Let (X, d) be a complete metric space endowed with a graph G and let f be a self mapping on X and $(\varphi, \phi, \Psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$. Suppose that the following is satisfied*

1. *for all $x, y \in X$, $(x, y) \in E(G)$ then $(fx, fy) \in E(G)$,*
2. *there exists $x_0 \in X$ such that $(x_0, fx_0) \in E(G)$,*
3. *G is transitive and has the property P,*
4. *for all $(x, y) \in E(G)$*

$$\int_0^{d(fx,fy)} \varphi(t)\mathbf{dt} \le \alpha(M(x, y)) \int_0^{M(x,y)} \varphi(t)\mathbf{dt},$$

where $M(x, y) = g(d(x, y), d(x, fx), d(y, fy))$.

then f has a fixed point. Also suppose that G_f is weakly connected, then f has a unique fixed point, where $v(G_f) = Fixf$ and $E(G_f) \subseteq E(G)$.

Proof. Let $x_{n+1} = fx_n$. Since $(x_0, x_1) \in E(G)$, so $(fx_0, fx_1) \in E(G)$ that is $(x_1, x_2) \in E(G)$. By this we have $(x_n, x_{n+1}) \in E(G)$. Since G is transitive we deduce that $(x_{m_k}, x_{n_k}) \in E(G)$ for $m_k, n_k \in \mathbf{N}$. Similar to the proof of theorem 2.2, we can show that $\{x_n\}$ is a cauchy sequence. From completeness of (X, d), there exists $a \in X$ such that $x_n \to a$. From property P, there exists $\{x_{n_k}\}$ such that $(x_{n_k}, a) \in E(G)$. So

$$\int_0^{d(fx_{n_{k+1}}, fa)} \varphi(t)\mathbf{dt} \le \alpha(M(x_{n_k}, a)) \int_0^{M(x_{n_k}, a)} \varphi(t)\mathbf{dt},$$

where $M(x_{n_k}, a) = g(d(x_{n_k}, a), d(a, fa), d(x_{n_k}, fx_{n_k}))$. Similar to the proof of theorem 2.2, we show that $fa = a$.

Uniqueness: Suppose that the subgraph G_f is weakly connected. Let $a, b \in Fixf$ with $a \neq b$.

So there exists a path $\{x_i\}_{i=0}^{l}$ of $l+1$ vertices with $x_0 = a$ and $x_l = b$ and $(x_i, x_{i+1}) \in E(\bar{G}_f)$ that is $(a, b) \in E(\bar{G}_f)$. Then

$$\int_0^{d(a,b)} \varphi(t)\mathbf{dt} = \int_0^{d(fa,fb)} \varphi(t)\mathbf{dt} \leq \alpha(M(a,b)) \int_0^{M(a,b)} \varphi(t)\mathbf{dt},$$

where $M(a, b) = g(d(a, b), d(a, fa), d(b, fb))$. Hence

$$\int_0^{d(a,b)} \varphi(t)\mathbf{dt} \leq \alpha(g(d(a,b),0,0)) \int_0^{g(d(a,b),0,0)} \varphi(t)\mathbf{dt},$$

from definition 2.1, $d(a, b) = 0$. That is $a = b$. □

Corollary 3.2. *Let $(X, d, \preceq)$ be a complete partially ordered metric space and let f be a self-mapping on X. Suppose that*

(1) *f is nondecreasing,*
(2) *there exists $x_0 \in X$ such that $x_0 \preceq fx_0$,*
(3) *for all $x, y \in X$ with $x \preceq y$ and $\varphi \in \Phi_1$ and $\alpha : \mathbf{R}^+ \to [0, 1)$ with $\limsup_{s\to t} \alpha(s) < 1, \quad t > 0$*
$$\int_0^{d(fx,fy)} \varphi(t)\mathbf{dt} \leq \alpha(M(x,y)) \int_0^{M(x,y)} \varphi(t)\mathbf{dt},$$
where $M(x, y) = g(d(x, y), d(x, fx), d(y, fy))$.
(4) *if nondecreasing sequence $\{x_n\}$ convergence to x, then $x_n \preceq x$.*

Then f has a fixed point.

Proof. Define a graph G by $v(G) = X$ and $E(G) = \{(x, y) \in X \times X; x \preceq y\}$. So all conditions of Theorem 3.1 is satisfied and then f has a fixed point. □

4. Application

In this section, we apply Theorem 2.2, to show the existence for the solution of the functional equations arising in dynamic programming.
Throughout this section, suppose that $(X, ||.||)$ and $(Y, ||.||')$ be real banach spaces and $S \subseteq X$ be the state space and $D \subseteq Y$ be the decision space. Assume that $B(S)$ be the set of all real-valued bounded functions on S. We equip $B(S)$ with the metric $d(f, g) = \sup_{x\in S} |f(x)-g(x)|$ for $f \in B(S)$. Note that $(B(S), d(f, g))$ is a Banach space. Consider the following functional equation

$$f(x) = \inf opt_{y\in D}\{A(x, y, f(a(x, y))), q(x, y) + B(x, y, f(b(x, y)))\}, \quad x \in S \tag{4.1}$$

where *opt* stands for sup or inf and a, b shows the transformation of the process and $f(x)$ presents the optimal return function with initial state x. Now, we mention two lemmas that is important in the proof of our main theorem.

Lemma 4.1. [11] *let C be a set, p and $q : C \to \mathbf{R}$ be mappings such that $opt_{y\in C}p(y)$, $opt_{y\in C}q(y)$ and $\sup_{y\in C}|p(y)-q(y)|$ are bounded. Then*

$$|opt_{y\in C}p(y) - opt_{y\in C}q(y)| \leq \sup_{y\in C}|p(y)-q(y)|.$$

Lemma 4.2. [11] *Let a, b, c and d be in $\mathbf{R}$. Then*

$$|opt\{a,b\} - opt\{c,d\}| \leq \max\{|a-c|, |b-d|\}.$$

Theorem 4.3. *Let $u, v : S \to \mathbf{R}$, $A, B : S\times D\times\mathbf{R} \to \mathbf{R}$ and $T : S\times D \to S$ satisfy the following conditions*

(1) *q, A, B are bounded,*
(2) $\int_0^{|A(x,y,u(T(x,y)))-A(x,y,v(T(x,y)))|}\varphi(t)\mathbf{d}t \leq \alpha(M(u,v))\int_0^{M(u,v)}\varphi(t)\mathbf{d}t$ *or*
$$\int_0^{|B(x,y,u(T(x,y)))-B(x,y,v(T(x,y)))|}\varphi(t)\mathbf{d}t \leq \alpha(M(u,v))\int_0^{M(u,v)}\varphi(t)\mathbf{d}t,$$
where $M(u,v) = g(d(u,v), d(u,Hu), d(v,Hv))$ and $Hz(x) = \inf opt_{y\in D}\{A(x,y,z(T(x,y))), q(x,y) + B(x,y,z(T(x,y)))\}$ for all $x \in S$ and $z \in B(S)$.

Then the functional equation 4.1 *has a unique solution.*

Proof. From assumption 1, there exists $k > 0$ such that

$$\sup\{|A(x,y,t)|, |B(x,y,t)|, |q(x,y)|\} < k, \quad (x,y,t)\in S\times D\times\mathbf{R}, \tag{4.2}$$

also from 4.2 and lemma 4.1, we deduce that H is the mapping on $B(S)$. By theorem 12.34 in [14], we obtain that for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$\int_C \varphi(t)\mathbf{d}t < \epsilon, \quad \forall C \subseteq [0, 2M] \quad \text{with} \quad m(C) \leq \delta \tag{4.3}$$

where $m(C)$ is the Lebesgue measure of C. Put

$$c(x,y,z) = opt_{y\in D}\{A(x,y,z(T(x,y))), q(x,y) + B(x,y,z(T(x,y)))\}$$

Let $x \in S$. For any $u, v \in B(S)$, there exists $y, h \in D$ with

$$Hu(x) \leq c(x,y,u), \quad Hu(x) > c(x,y,u) - \delta$$
$$Hv(x) \leq c(x,y,v), \quad Hv(x) > c(x,y,v) - \delta,$$

so by using lemma 4.2, we have

$$\begin{aligned} Hu(x) - Hv(x) &< c(x,y,u) - c(x,h,v) + \delta \\ &< \max\{|A(x,y,u) - A(x,h,v)|, |B(x,y,u) - B(x,h,v)| + \delta, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} Hu(x) - Hv(x) &> c(x,y,u) - c(x,h,v) - \delta \\ &> -\max\{|A(x,y,u) - A(x,h,v)|, |B(x,y,u) - B(x,h,v)| - \delta. \end{aligned} \tag{4.5}$$

Hence

$$|Hu(x) - Hv(x)| < \max\{|A(x,y,u) - A(x,h,v)|, |B(x,y,u) - B(x,h,v)| + \delta, \tag{4.6}$$

put

$$A_1 = A(x,y,u), \quad A_2 = A(x,h,v), \quad B_1 = B(x,y,u), \quad B_2 = B(x,h,v),$$

so assumption 2 combining with 4.6 and 4.3 implies that

$$\begin{aligned}
\int_0^{|Hu(x)-Hv(x)|} \varphi(t)\mathbf{d}t &\leq \int_0^{\max\{|A_1-A_2|,|B_1-B_2|\}+\delta} \varphi(t)\mathbf{d}t \\
&= \max\{\int_0^{|A_1-A_2|+\delta} \varphi(t)\mathbf{d}t, \int_0^{|B_1-B_2|+\delta} \varphi(t)\mathbf{d}t\} \\
&= \max\{\int_0^{|A_1-A_2|} \varphi(t)\mathbf{d}t + \int_{|A_1-A_2|}^{|A_1-A_2|+\delta} \varphi(t)\mathbf{d}t, \int_0^{|B_1-B_2|} \varphi(t)\mathbf{d}t + \int_{|B_1-B_2|}^{|B_1-B_2|+\delta} \varphi(t)\mathbf{d}t\} \\
&\leq \max\{\int_0^{|A_1-A_2|} \varphi(t)\mathbf{d}t, \int_0^{|B_1-B_2|} \varphi(t)\mathbf{d}t\} + \max\{\int_{|A_1-A_2|}^{|A_1-A_2|+\delta} \varphi(t)\mathbf{d}t, \int_{|B_1-B_2|}^{|B_1-B_2|+\delta} \varphi(t)\mathbf{d}t\} \\
&\leq \alpha(M(u,v)) \int_0^{M(u,v)} \varphi(t)\mathbf{d}t + \epsilon, \qquad (4.7)
\end{aligned}$$

letting $\epsilon \to 0^+$ implies that

$$\int_0^{d(Hu(x),Hv(x))} \varphi(t)\mathbf{d}t \leq \alpha(M(u,v)) \int_0^{M(u,v)} \varphi(t)\mathbf{d}t.$$

Thus by theorem 2.2, the equation has a solution. □

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