ASIAN JOURNAL OF MATHEMATICS AND APPLICATIONS
Volume 2013, Article ID ama0054, 8 pages
ISSN 2307-7743
http://scienceasia.asia

A FINITE REFLECTION FORMULA FOR A POLYNOMIAL APPROXIMATION TO THE RIEMANN ZETA FUNCTION

STEPHEN CROWLEY

ABSTRACT. The Riemann zeta function can be written as the Mellin transform of the unit interval map $w(x) = \lfloor x^{-1} \rfloor \left(x \lfloor x^{-1} \rfloor + x - 1 \right)$ multiplied by $s\frac{s+1}{s-1}$. A finite-sum approximation to $\zeta(s)$ denoted by $\zeta_w(N;s)$ which has real roots at $s=-1$ and $s=0$ is examined and an associated function $\chi(N;s)$ is found which solves the reflection formula $\zeta_w(N;1-s) = \chi(N;s)\,\zeta_w(N;s)$. A closed-form expression for the integral of $\zeta_w(N;s)$ over the interval $s = -1 \ldots 0$ is given. The function $\chi(N;s)$ is singular at $s=0$ and the residue at this point changes sign from negative to positive between the values of $N=176$ and $N=177$. Some rather elegant graphs of $\zeta_w(N;s)$ and the reflection functions $\chi(N;s)$ are also provided. The values $\zeta_w(N;1-n)$ for integer values of n are found to be related to the Bernoulli numbers.

Contents

1. The Riemann Zeta Function as the Mellin Transform of a Unit Interval Map

The Riemann zeta function can be written as the Mellin transform of the unit interval map $w(x) = \lfloor x^{-1} \rfloor \left(x \lfloor x^{-1} \rfloor + x - 1 \right)$ multiplied by $s\frac{s+1}{s-1}$. [3][2]

$$
\begin{aligned}
\zeta_w(s) &= \zeta(s) \forall -s \notin \mathbb{N}^* \\
&= s\tfrac{s+1}{s-1} \int_0^1 \lfloor x^{-1} \rfloor \left(x \lfloor x^{-1} \rfloor + x - 1 \right) x^{s-1} \mathrm{d}x \\
&= s\tfrac{s+1}{s-1} \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} n(xn + x - 1) x^{s-1} \mathrm{d}x \\
&= \sum_{n=1}^{\infty} s\tfrac{s+1}{s-1} \left(-\tfrac{n^{1-s} - n(n+1)^{-s} - sn^{-s}}{s(s+1)} \right) \\
&= \sum_{n=1}^{\infty} \tfrac{n(n+1)^{-s} - n^{1-s} + sn^{-s}}{s-1} \\
&= \tfrac{1}{s-1} \sum_{n=1}^{\infty} n(n+1)^{-s} - n^{1-s} + sn^{-s}
\end{aligned} \tag{1}
$$

2010 *Mathematics Subject Classification.* 14G10.
Key words and phrases. Riemann's zeta function.

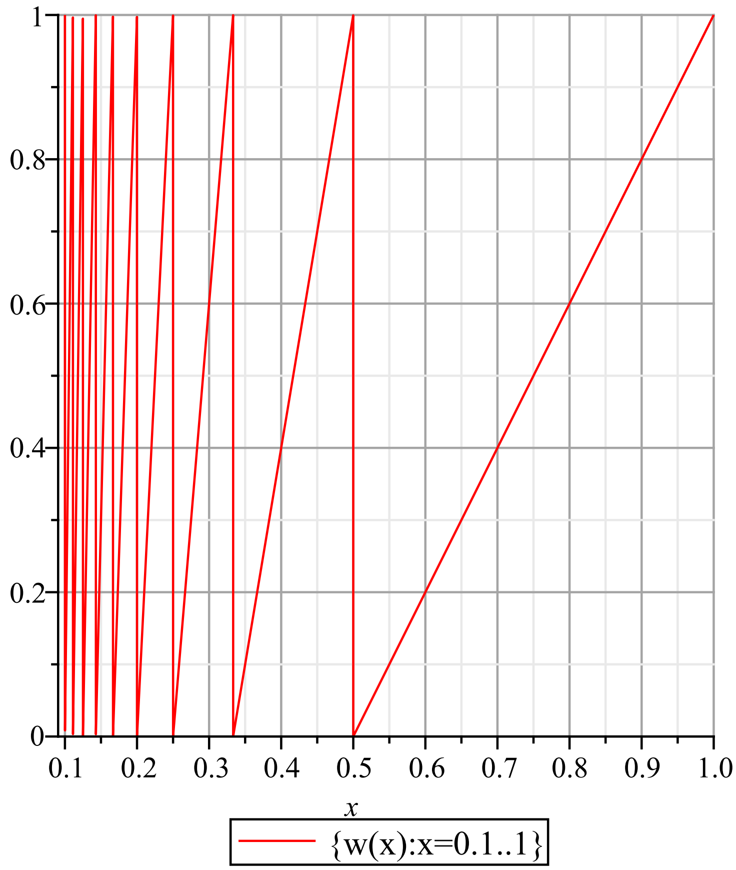

FIGURE 1. The Harmonic Sawtooth map

1.1. **The Truncated Zeta Function.** The substition $\infty \to N$ is made in the infinite sum appearing the expression for $\zeta_w(s)$ to get a finite polynomial approximation

$$\begin{aligned} \zeta_w(N;s) &= \tfrac{1}{s-1}\sum_{n=1}^{N} n(n+1)^{-s} - n^{1-s} + sn^{-s} \\ &= \tfrac{1}{s-1}\left(s + (N+1)^{1-s} - 1 + s\sum_{n=2}^{N} n^{-s} - \sum_{n=2}^{N+1} n^{-s}\right) \\ &= \tfrac{N}{(s-1)(N+1)^s} - \tfrac{\cos(\pi s)\Psi(s-1,N+1)}{\Gamma(s)} + \zeta\left(s\right)\forall s \in \mathbb{N}^* \end{aligned} \quad (2)$$

with equality in the limit except at the negative integers

$$\lim_{N\to\infty} \zeta_w(N;s) \quad = \zeta(s)\forall - s \notin \mathbb{N}^* \quad (3)$$

and where $\Psi\left(x,n\right) = \frac{\mathrm{d}}{\mathrm{d}x^n}\Psi\left(x\right)$ is the polygamma function and $\Psi\left(x\right) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\Gamma\left(x\right)\right)$ is the digamma function. The functions $\zeta_w\left(N;s\right)$ have real zeros at $s = -1$ and $s = 0$, that is

$$\lim_{s\to -1} \zeta_w\left(N;s\right) = \lim_{s\to 0} \zeta_w\left(N;s\right) = 0 \quad (4)$$

One possible idea is that the functions $\zeta_w(N;s)$ can be orthonormalized over the interval $s = -1\dots0$ via the Gram-Schmidt process[4] and that the result might possibly shed some light on the zeroes of $\zeta(s)$. Let the logarithmic integral be defined

$$\mathrm{Li}(x) \quad = \quad \int_0^{\ln(x)} \frac{e^y - 1}{y}\mathrm{d}y + \ln(\ln(x)) + \gamma \tag{5}$$

where $\gamma = 0.57721\dots$ is Euler's constant, then the normalization factors are given by the integral

$$\begin{aligned} \int_{-1}^{0} \zeta_w(N;s)\,\mathrm{d}s &= \int_{-1}^{0} \sum_{n=1}^{N} \frac{n(n+1)^{-s} - n^{1-s} + sn^{-s}}{s-1}\mathrm{d}s \\ &= 1 + \frac{N}{N+1}\left(\mathrm{Li}(N+1) - \mathrm{Li}\left((N+1)^2\right)\right) + \sum_{n=1}^{N-1} \frac{n}{\ln(n+1)} \end{aligned} \tag{6}$$

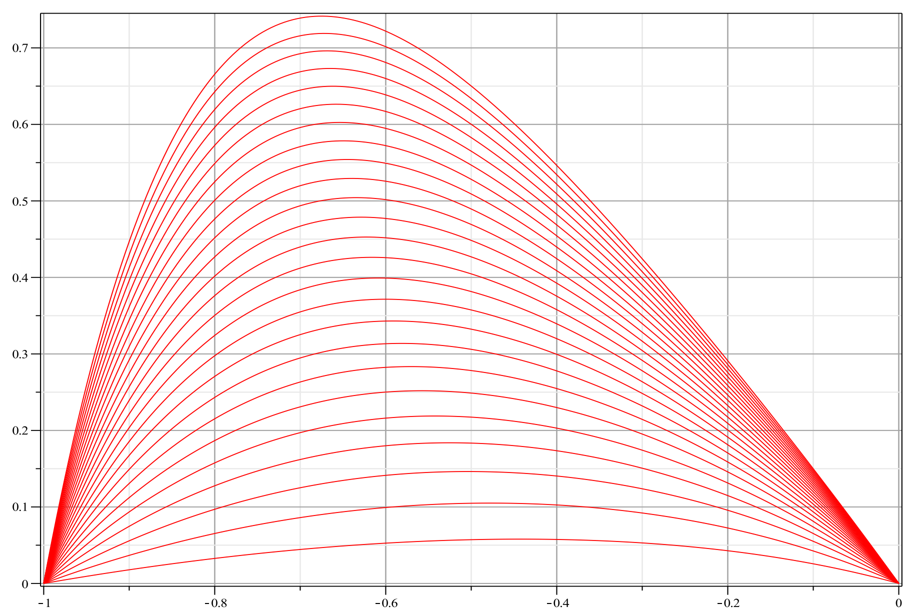

FIGURE 2. $\{\zeta_w(N;s) : s = -1\dots0, N = 1\dots25\}$

The following table lists the values of $\zeta_w(N;1-n)$ for $n = 2\dots12$.

$$\left[\begin{array}{c} 0 \\ -\frac{1}{6}N - \frac{1}{6}N^2 \\ -\frac{1}{4}N - \frac{1}{2}N^2 - \frac{1}{4}N^3 \\ -\frac{7}{30}N - \frac{4}{5}N^2 - \frac{13}{15}N^3 - \frac{3}{10}N^4 \\ -\frac{1}{6}N - \frac{11}{12}N^2 - \frac{5}{3}N^3 - \frac{5}{4}N^4 - \frac{1}{3}N^5 \\ -\frac{5}{42}N - \frac{6}{7}N^2 - \frac{97}{42}N^3 - \frac{20}{7}N^4 - \frac{23}{14}N^5 - \frac{5}{14}N^6 \\ -\frac{1}{8}N - \frac{19}{24}N^2 - \frac{21}{8}N^3 - \frac{14}{3}N^4 - \frac{35}{8}N^5 - \frac{49}{24}N^6 - \frac{3}{8}N^7 \\ -\frac{13}{90}N - \frac{8}{9}N^2 - \frac{26}{9}N^3 - \frac{56}{9}N^4 - \frac{371}{45}N^5 - \frac{56}{9}N^6 - \frac{22}{9}N^7 - \frac{7}{18}N^8 \\ -\frac{1}{10}N - \frac{21}{20}N^2 - \frac{18}{5}N^3 - \frac{79}{10}N^4 - \frac{63}{5}N^5 - \frac{133}{10}N^6 - \frac{42}{5}N^7 - \frac{57}{20}N^8 - \frac{2}{5}N^9 \\ -\frac{1}{66}N - \frac{10}{11}N^2 - \frac{101}{22}N^3 - \frac{120}{11}N^4 - \frac{199}{11}N^5 - \frac{252}{11}N^6 - \frac{221}{11}N^7 - \frac{120}{11}N^8 - \frac{215}{66}N^9 - \frac{9}{22}N^{10} \\ -\frac{1}{12}N - \frac{1}{2}N^2 - \frac{55}{12}N^3 - \frac{121}{8}N^4 - \frac{55}{2}N^5 - \frac{110}{3}N^6 - \frac{77}{2}N^7 - \frac{231}{8}N^8 - \frac{55}{4}N^9 - \frac{11}{3}N^{10} - \frac{5}{12}N^{11} \end{array}\right]$$

1.1.1. *Integrating Over the Critical Strip.* There is a formula similiar to (6) which gives the integral of $\zeta_w(N;s)$ over the critical strip $0 \leqslant \mathrm{Re}(s) \leqslant 1$.

$$\begin{aligned} \int_0^1 \zeta_w(N;c+is)\,\mathrm{d}c &= 1+\tfrac{N}{N+1}\left(\mathrm{Ei}_1(is\ln(N+1)-\ln(N+1))-\mathrm{Ei}_1(is\ln(N+1))\right) \\ +\sum_{n=1}^{N-1}\tfrac{n(n+1)^{-is}}{(n+1)\ln(n+1)} \end{aligned} \tag{7}$$

where $\mathrm{Ei}_1(t)$ is the exponential integral defined by

$$\mathrm{Ei}_1(t) = t\int_0^1\int_0^1 e^{-txy}\mathrm{d}y\mathrm{d}x - \gamma - \ln(t) \tag{8}$$

The contribution from the Ei term vanishes as $s \to \infty$, that is

$$\lim_{s\to\infty}\frac{N}{N+1}\left(\mathrm{Ei}_1(is\ln(N+1)-\ln(N+1))-\mathrm{Ei}_1(is\ln(N+1))\right)=0 \tag{9}$$

1.1.2. *The Reflection Formula.* There is a reflection equation for the finite-sum approximation $\zeta_w(N;s)$ which is similiar to the well-known formula $\zeta(1-s)=\chi(s)\zeta(s)$ with

$$\chi(s) = 2(2\pi)^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s).$$

The solution to

$$\zeta_w(N;1-s) = \chi(N;s)\zeta_w(N;s) \tag{10}$$

is given by the expression

$$\begin{aligned} \chi(N;s) &= \frac{\zeta_w(N;1-s)}{\zeta_w(N;s)} \\ &= \frac{\sum_{n=1}^{N} - \frac{-n^s+(n+1)^{s-1}n+n^{s-1}-n^{s-1}s}{s}}{\sum_{n=1}^{N}\frac{-n^{1-s}+(n+1)^{-s}n+n^{-s}s}{s-1}} \\ &= -\frac{(s-1)\sum_{n=1}^{N}-n^s+(n+1)^{s-1}n+n^{s-1}-n^{s-1}s}{s\sum_{n=1}^{N}-n^{1-s}+(n+1)^{-s}n+n^{-s}s} \end{aligned} \tag{11}$$

which satisfies

$$\chi(N;1-s) = \chi(N;s)^{-1} \tag{12}$$

The functions $\chi(N;s)$, indexed by N, have singularities at $s=0$. Let

$$\begin{aligned} a(N) &= \sum_{n=1}^{N} n(\ln(n+1)-\ln(n)) \\ b(N) &= \sum_{n=1}^{N}\frac{\ln(n)n^2-\ln(n+1)n^2-\ln(n)}{n(n+1)} \\ c(N) &= \frac{1}{2}\sum_{n=1}^{N} n\left(\ln(n+1)^2-\ln(n)^2\right) \end{aligned} \tag{13}$$

then the residue at the singular point $s=0$ is given by the expression

(14)

$$\begin{aligned} \operatorname*{Res}_{s=0}(\chi(N;s)) &= -\operatorname*{Res}_{s=1}(\chi(N;s)^{-1}) \\ &= \frac{1+\gamma+\Psi(n+2)-\frac{2}{N+1}+b(N)-\frac{N(\ln(\Gamma(N+1))-c(N))}{(N-a(N))(N+1)}}{a(N)-N} \\ &= \frac{1+\gamma+\Psi(n+2)-\frac{2}{N+1}+\sum_{n=1}^{N}\frac{\ln(n)n^2-\ln(n+1)n^2-\ln(n)}{n(n+1)}-\frac{N\left(\ln(\Gamma(N+1))-\frac{1}{2}\sum_{n=1}^{N}n\left(\ln(n+1)^2-\ln(n)^2\right)\right)}{\left(N-\sum_{n=1}^{N}n(\ln(n+1)-\ln(n))\right)(N+1)}}{\left(\sum_{n=1}^{N}n(\ln(n+1)-\ln(n))\right)-N} \end{aligned}$$

which has the limit

$$\lim_{N\to\infty}\operatorname*{Res}_{s=0}(\chi(N;s)) = 1 \tag{15}$$

We also have the residue of the reciprocal at $s = 2$

$$\operatorname*{Res}_{s=2}(\chi(N;s)^{-1}) = \frac{\frac{2N}{(N+1)^2} - 2\Psi(1,N+1) + 2\zeta(2)}{\frac{(N+1)^2}{2} - \frac{N}{2} - \frac{1}{2} - \sum_{n=1}^{N} n(\ln(n+1) + \ln(n+1)n - \ln(n) - n\ln(n))} \tag{16}$$

which vanishes as N tends to infinity

$$\lim_{N\to\infty} \operatorname*{Res}_{s=2}(\chi(N;s)^{-1}) = 0 \tag{17}$$

As can be seen in the figures below, the residue at $s = 0$ changes sign from negative to positive between the values of $N = 176$ and $N = 177$.

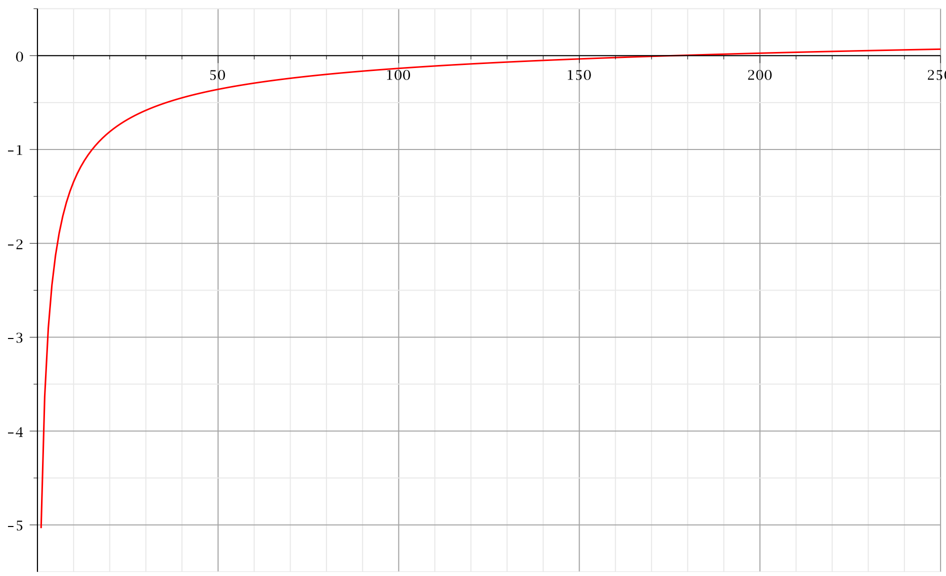

FIGURE 3. $\left\{\operatorname*{Res}_{s=0}(\chi(N;s)) : N = 1\ldots 250\right\}$

For any positive integer N, we have the limits

$$\begin{aligned} \lim_{s\to 0} \chi(N;s) &= \infty \\ \lim_{s\to 0} \frac{\mathrm{d}^n}{\mathrm{d}s^n}\chi(N;s) &= \infty \\ \lim_{s\to \frac{1}{2}} \chi(N;s) &= 1 \\ \lim_{s\to 1} \chi(N;s) &= 0 \\ \lim_{s\to 2} \chi(N;s) &= 0 \\ \lim_{s\to 1} \frac{\mathrm{d}}{\mathrm{d}s}\chi(N;s) &= 0 \end{aligned} \tag{18}$$

The line $\operatorname{Re}(s) = \frac{1}{2}$ has a constant modulus

$$\left|\chi\left(N; \frac{1}{2} + is\right)\right| = 1 \tag{19}$$

There is also the complex conjugate symmetry

$$\chi(N; x+iy) = \overline{\chi(N; x-iy)} \tag{20}$$

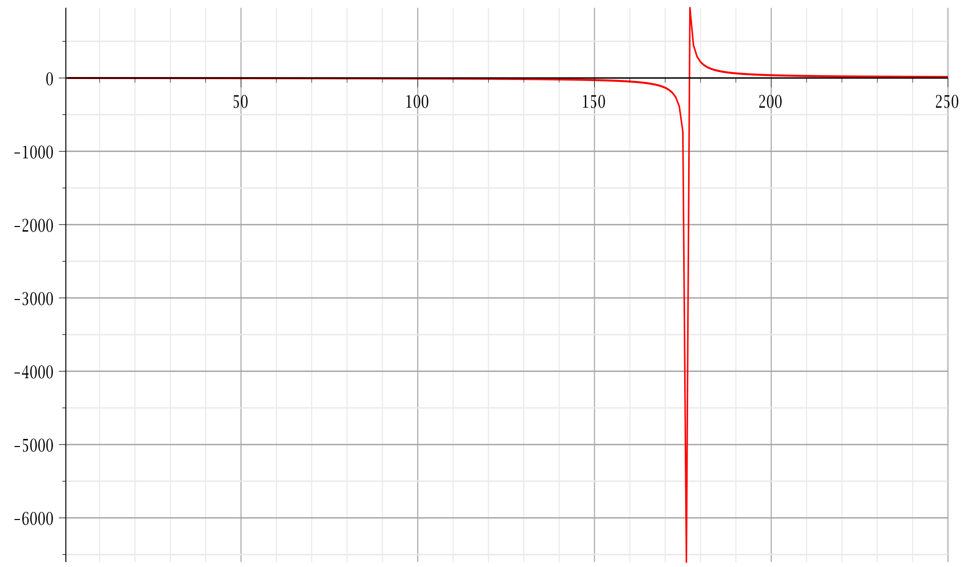

FIGURE 4. $\left\{\operatorname*{Res}_{s=0}(\chi(N;s))^{-1} : N = 1\ldots 250\right\}$

If $s = n \in \mathbb{N}^*$ is a positive integer then $\chi(N;n)$ can be written as

$$\begin{aligned}\chi(N;n) &= \frac{\zeta_w(N;1-n)}{\zeta_w(N;n)} \\ &= \frac{\sum_{m=1}^{N} - \sum_{k=1}^{n-2} \frac{m^k}{n}\binom{n-1}{k-1}}{\frac{N}{(n-1)(N+1)^n} - \frac{\cos(\pi n)\Psi(n-1,N+1)}{\Gamma(n)} + \zeta(n)} \\ &= \frac{-\sum_{m=1}^{N} \frac{1}{n}\left((n-1)m^{n-1} + m^n - (m+1)^{n-1}m\right)}{\frac{N}{(n-1)(N+1)^n} - \frac{\cos(\pi n)\Psi(n-1,N+1)}{\Gamma(n)} + \zeta(n)}\end{aligned} \tag{21}$$

where $\binom{n-1}{k-1}$ is of course a binomial. The Bernoulli numbers[1] make an appearance since

$$\chi(N;2n)\,\zeta_w(N;2n) = B_{2n}(N+1)^2\,\frac{(2n+1)}{2} + \ldots \tag{22}$$

The denominator of $\chi(N;n)$ has the limits

$$\begin{aligned}\lim_{N\to\infty}\zeta_w(N;n) &= \zeta(n) \\ \lim_{n\to\infty}\zeta_w(N;n) &= 1\end{aligned} \tag{23}$$

Another interesting formula gives the limit at $s = 1$ of the quotient of successive functions

$$\begin{aligned}\lim_{s=1}\frac{\chi(N+1;s)}{\chi(N;s)} &= \frac{(N+2)N(N+1-a(N+1))}{(N+1)^2(N-a(N))} \\ &= \frac{(N+2)N\left(N+1-\sum_{n=1}^{N+1} n(\ln(n+1)-\ln(n))\right)}{(N+1)^2\left(N-\sum_{n=1}^{N} n(\ln(n+1)-\ln(n))\right)}\end{aligned} \tag{24}$$

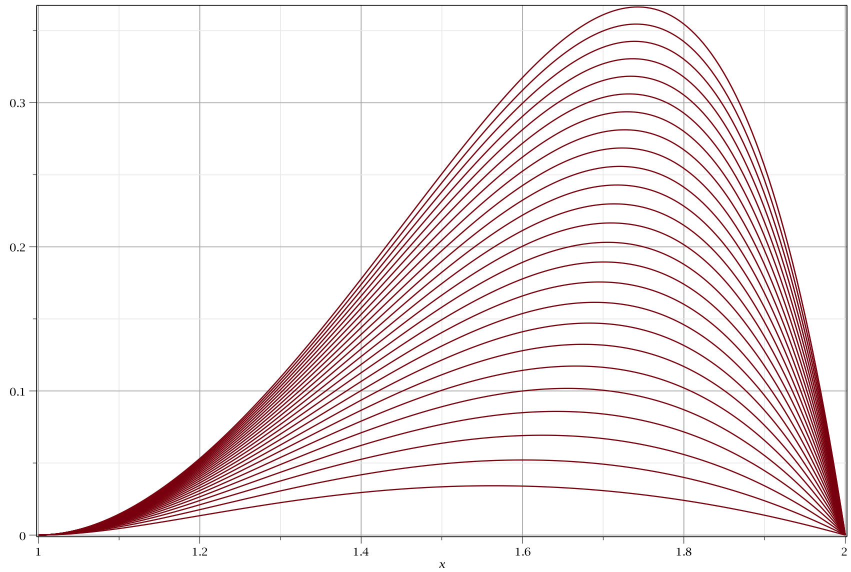

FIGURE 5. $\{\chi(N;s) : s = 1 \ldots 2, N = 1 \ldots 25\}$

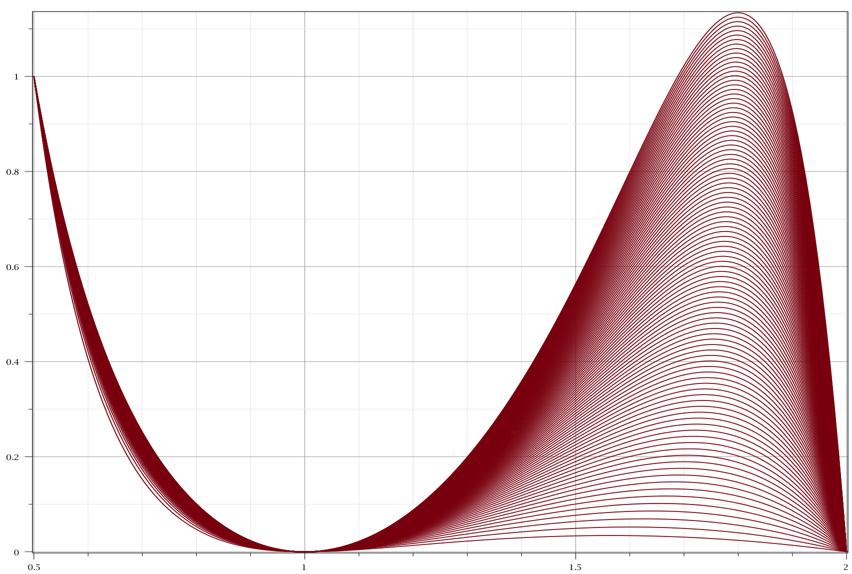

FIGURE 6. $\left\{\chi(N;s) : s = \frac{1}{2} \ldots 2, N = 1 \ldots 100\right\}$

Let

$$\nu(s) = \chi(\infty; s) = \frac{\zeta(1-s)}{\zeta(s)} \tag{25}$$

Then the residue at the even negative integers is

$$\operatorname*{Res}_{s=-n}(\nu(s)) = \begin{cases} \frac{\zeta(1-n)}{\frac{\mathrm{d}}{\mathrm{d}s}\zeta(s)|_{s=-n}} & n\,\text{even} \\ 0 & n\,\text{odd} \end{cases} \tag{26}$$

References

[1] G Arfken. *Mathematical Methods for Physicists, 3rd ed.*, chapter 5.9, Bernoulli Numbers, Euler-Maclaurin Formula., pages 327–338. Academic Press, 1985.

[2] Stephen Crowley. Integral Transforms of the Harmonic Sawtooth Map, The Riemann Zeta Function, Fractal Strings, and a Finite Reflection Formula http://arxiv.org/abs/1210.5652, October 2012.

[3] Stephen Crowley. Two new zeta constants: Fractal string, continued fraction, and hypergeometric aspects of the riemann zeta function. http://arxiv.org/abs/1207.1126, July 2012.

[4] G. H. Golub and C. F. Van Loan. *Matrix Computations*. Johns Hopkins, 3 edition, 1996.

stephen.crowley@mavs.uta.edu