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AN ANALYTICAL SOLUTION FOR THE PERTURBED RESTRICTED THREE-BODY PROBLEM USING VARIATION OF PARAMETERS METHOD

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Abstract: In this work, the equations of motion of the restricted three-body problem under the effects of the oblateness of less massive primary and the radiation pressure of the bigger massive primary are expressed. The analytical solution is obtained by the variation of parameters method. The locations of the libration points are obtained. The periodic orbits around each of these points are investigated for the Sun-Earth system, the zero-velocity curves, the phase spaces, and the Poincare surface sections are presented for one of the collinear libration points and one of the non-linear libration points. The obtained results are compared with previous works such as Kunitsyn[13] and Schuerman[14], and good agreements with these results are found.

Introduction

The variation of parameters method is one of the important methods of theoretical and applied mathematics. The applications of this method are concerned mainly with the determination of maximum and minimum of certain expressions involving unknown functions. For orbital dynamics this method was introduced by the Swiss-born mathematician Leonhard Euler (1707 - 1783) and completed by the Italian-French mathematician Joseph-Louis Lagrange (1736 - 1813) Abell [1]and Efroimsky[2] Lagrange had put a reduced two-body problem as an unperturbed solution and had assumed that all perturbations come from the gravitational force. In the 20th century Vallado[3], celestial mechanics began to consider interactions that depend on both positions and velocities (relativistic corrections, atmospheric drag, inertial forces). Therefore, the method of

Key words and phrases: Method of Variation of Parameters; Periodic Orbits; Oblateness; Radiation Pressure; The Restricted Three-Body Problem.

variation of parameters used by Lagrange was extended to the situation with velocity-dependent forces Moulton [4]. Geometrically, this method is a representation of an orbit as a set of points, each of which is contributed by a member of some chosen family of curves C(K), where K stands for a set of constants that number a particular -curve within the family curves Lovett[5]

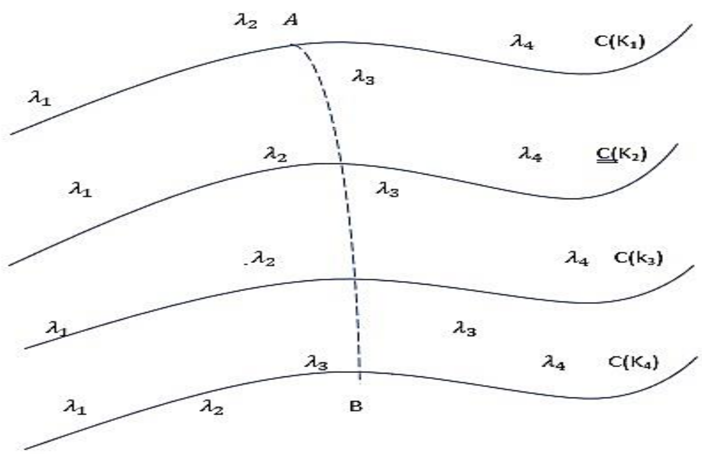


Fig.1. Each point of the orbit is contributed by a member of some family of curves C(K) of a certain type, K standing for a set of constants that member a particular curve within the family. Motion from A to B is, first, due to the motion along the curve C(K) from λ_1 *to* λ_2 and, second, due to the fact that during this motion the curve itself was evolving from C (K_1) to C (K_2).

This situation is depicted within the family of curves Fig.1.Point A of the orbit coincides with some point$\lambda_1$on a curve $C\,(K_1)$. Point B of the orbit coincides with point$\lambda_2$on some other curve$C\,(K_2)$ of the same family, etc. This way, orbital motion from A to B becomes a superposition of motion along $C_K$from$\lambda_1$to$\lambda_2$and a gradual distortion of the curve $C_K$from the shape $C\,(K_1)$ to the shape $C\,(K_2)$. Normally the curves $C_K$are chosen to be ellipses or hyperbolae to be managed analyzed Oberti[6]. Many analytical theories depend on the central body and the perturbs force, therefore it is needed to study the problem with many perturbing forces Vallado[7].

Therefore, the method of variation of parameters is useful to study the motion of a massless particle in restricted three-body problem which considered as one of the most important objects in astro-dynamics. There are many studies which treated the restricted three-body problem under different perturbing forces Srivastava[8]. But most important perturbing forces are the radiation pressure and the oblateness of the primaries Efroimsky [2], George [9], Douskos[10], Sharma[11], Simmons[12], Kunitsyn[13] and Schuerman[14].

In this work, the equations of motion of the restricted three-body problem in the classical form are presented under the effects of the radiation pressure of the more massive body and the oblateness of less massive body. The variation of parameters method is used to obtain the analytical solution. From this solution, the libration points are obtained and the stability around each point is studied.

Equations of Motion

Using a barycentric-synodic coordinate system (X, Y, Z) and dimensionless variables, the equations of motion of a test particle in the circular restricted three-body problem under the effects of the oblateness of the small primary and the radiation pressure of the bigger primary can be expressed as

$$\ddot{X} - 2n\dot{Y} = U_X \qquad (1.1)$$

$$\ddot{Y} + 2n\dot{X} = U_Y \qquad (1.2)$$

$$\ddot{Z} = U_Z \qquad (1.3)$$

Since the above system is rotating around the Z axis by a constant angular velocity, then Z = constant, so the system of Equations (1) is reduced to the system

$$\ddot{X} - 2n\dot{Y} = U_X \qquad (2.1)$$

$$\ddot{Y} + 2n\dot{X} = U_Y \qquad (2.2)$$

Where

$$U = \frac{n^2}{2}(X^2 + Y^2) + \frac{(1-\beta)(1-\mu)}{r_1} - \frac{\mu}{r_2} - \frac{\mu A}{2r_2^3} \qquad (3)$$

Where

$$r_1 = \sqrt{(\mathrm{x} - \mu)^2 + y^2}, \qquad r_2 = \sqrt{(\mathrm{x} + 1 - \mu)^2 + y^2}$$

$$\mathrm{U}_x = n^2\mathrm{x} - \frac{(1-\mu)(\mathrm{x}-\mu)\,(1-\beta)}{\mathrm{r}_1^3} - \frac{\mu(\mathrm{x}-\mu+1)}{\mathrm{r}_2^3} - \frac{3\,\mathrm{A}\,\mu(\mathrm{x}-\mu+1)}{2\,\mathrm{r}_2^5} \qquad (4)$$

$$\mathrm{U}_y = n^2 y - \frac{(1-\mu)(1-\beta)\,y}{r_1^3} - \frac{\mu y}{r_2^3} - \frac{3\,A\,\mu y}{2\,r_2^5} \qquad (5)$$

Where n is the mean motion of the smaller primary and given by

$n^2 = 1 + \frac{3}{2}\mathrm{A}$, where $A = \frac{r_e^2 - r_p^2}{5\,R^2}$ Ibrahim[15], r_e and r_p are the equatorial and polar radii of the oblate body, and R is the distance between the two primaries, is considered as unity Ismail[16] .

Since the radiation, pressure force F_r and the gravitational force F_g are acting on the body in opposite directions, so that

$$F_g - F_r = F_g(1-\beta) \tag{6}$$

Where

F_g and F_r are the gravitational force and radiation force respectively,

$\beta = \frac{F_r}{F_g}$. In dimensionless coordinates $m_1 + m_2 = 1$, $\mu = \frac{m_2}{m_1 + m_2}$ is the mass ratio of the system.

Method of Variation of Parameters

To solve the system of Equations (2) we use the method of variation of parameters Chen[17] and Palais[18], which needed to reduce the above system to first order, so that let $u = DX, Du = D^2X,\ v = DY, Dv = D^2Y$. Then the system of Equations (2) becomes

$$Du = 2nv + U_X \tag{7.1}$$

$$Dv = -2nu + U_Y \tag{7.2}$$

$$DX = u \tag{7.3}$$

$$DY = v \tag{7.4}$$

Equations (7) are system of ODEs, which expressed in the matrix form as

$$[\dot{\chi}(t)] = \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{X} \\ \dot{Y} \end{bmatrix} = \begin{bmatrix} 0 & 2n & 0 & 0 \\ -2n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ X \\ Y \end{bmatrix} + \begin{bmatrix} U_X \\ U_Y \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

Now, the homogenous and the particular- solutions for the system of Equations (8) well be obtained. At first the homogenous solution $\boldsymbol{\chi_H}$ is obtained when $U_X = 0$ and $U_Y = 0$; then

$$[\dot{\chi}(t)] = \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{X} \\ \dot{Y} \end{bmatrix} = \begin{bmatrix} 0 & 2n & 0 & 0 \\ -2n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{9}$$

The auxiliary equation for the homogenous system of Equations (9) is

$$|A - \lambda I| = \begin{vmatrix} 0-\lambda & 2n & 0 & 0 \\ -2\,n & 0-\lambda & 0 & 0 \\ 0 & 0 & 0-\lambda & 0 \\ 0 & 1 & 0 & 0-\lambda \end{vmatrix} = 0$$

The roots are $\boldsymbol{\lambda}_{1,2} = 0$; $\boldsymbol{\lambda}_{3,4} = \pm 2in$; the homogenous solution of system (9) is,

$$\chi_H = c_1\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + t\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right\} + c_2\begin{bmatrix}2n\\-2in\\i\\1\end{bmatrix}e^{-2int} + c_3\begin{bmatrix}2n\\2in\\-i\\1\end{bmatrix}e^{2int}$$

$$= c_1\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + t\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right\} + c_2\begin{bmatrix}2n\\-2in\\i\\1\end{bmatrix}(\cos 2nt - i\sin 2nt) + c_3\begin{bmatrix}2n\\2in\\-i\\1\end{bmatrix}(\cos 2nt + i\sin 2nt)$$

$$= c_1\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + t\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right\} + (c_2 + c_3)\left\{\begin{bmatrix}2n\\0\\0\\1\end{bmatrix}\cos 2nt + \begin{bmatrix}0\\-2n\\1\\0\end{bmatrix}\sin 2nt\right\}$$

$$+i\,(c_2 - c_3)\left\{\begin{bmatrix}0\\-2n\\1\\0\end{bmatrix}\cos 2nt + \begin{bmatrix}-2n\\0\\0\\-1\end{bmatrix}\sin 2nt\right\}$$

put $c_2 = (c_2 + c_3)$; $c_3 = I(c_2 - c_3)$; then

$$\chi_H = c_1\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + t\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right\} + c_2\left\{\begin{bmatrix}2n\\0\\0\\1\end{bmatrix}\cos 2nt + \begin{bmatrix}0\\-2n\\1\\0\end{bmatrix}\sin 2nt\right\} + c_3\left\{\begin{bmatrix}0\\-2n\\1\\0\end{bmatrix}\cos 2nt + \begin{bmatrix}-2n\\0\\0\\-1\end{bmatrix}\sin 2nt\right\} \quad (10)$$

Where c_1, c_2, and c_3 are arbitrary linear independent constants. Then substitute into the system of Equations (2) we get,

$$\chi_H = c_1\left\{\begin{bmatrix}0\\1\end{bmatrix} + t\begin{bmatrix}1\\0\end{bmatrix}\right\} + c_2\left\{\begin{bmatrix}0\\1\end{bmatrix}\cos 2nt + \begin{bmatrix}1\\0\end{bmatrix}\sin 2nt\right\} + c_3\left\{\begin{bmatrix}1\\0\end{bmatrix}\cos 2nt + \begin{bmatrix}0\\-1\end{bmatrix}\sin 2nt\right\} \quad (11)$$

Now to obtain the particular- solution χ_P, for the nonhomogeneous system (2) which is in the form,

$$\chi_p = A_1(t)\left\{\begin{bmatrix}0\\1\end{bmatrix} + t\begin{bmatrix}1\\0\end{bmatrix}\right\} + A_2(t)\left\{\begin{bmatrix}0\\1\end{bmatrix}\cos 2nt + \begin{bmatrix}1\\0\end{bmatrix}\sin 2nt\right\} + A_3(t)\left\{\begin{bmatrix}1\\0\end{bmatrix}\cos 2nt + \begin{bmatrix}0\\-1\end{bmatrix}\sin 2nt\right\} \quad (12)$$

Since, the particular- solution χ_P satisfied the system of Equations (2) then,

$$\dot{\chi}_p = \begin{bmatrix}0 & \alpha\\-\alpha & 0\end{bmatrix}\chi_p \quad (13)$$

where $2n\,i = \alpha$;

$$\dot{\chi}_p = \dot{A}_1(t)\left\{\begin{bmatrix}0\\1\end{bmatrix} + t\begin{bmatrix}1\\0\end{bmatrix}\right\} + A_1(t)\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\} + \dot{A}_2(t)\left\{\begin{bmatrix}0\\1\end{bmatrix}\cos\alpha t + \begin{bmatrix}1\\0\end{bmatrix}\sin\alpha t\right\} + A_2(t)\left\{\begin{bmatrix}0\\1\end{bmatrix}(-\alpha\sin\alpha t) + \begin{bmatrix}1\\0\end{bmatrix}\alpha\cos\alpha t\right\}$$
$$+ \dot{A}_3(t)\left\{\begin{bmatrix}1\\0\end{bmatrix}\cos\alpha t + \begin{bmatrix}0\\-1\end{bmatrix}\sin\alpha t\right\} + A_3(t)\left\{\begin{bmatrix}1\\0\end{bmatrix}(-\alpha\sin\alpha t) + \begin{bmatrix}0\\-1\end{bmatrix}\alpha\cos\alpha t\right\}$$

$$= \begin{bmatrix}0 & \alpha\\-\alpha & 0\end{bmatrix}.\left\{A_1(t)\left\{\begin{bmatrix}0\\1\end{bmatrix} + t\begin{bmatrix}1\\0\end{bmatrix}\right\} + A_2(t)\left\{\begin{bmatrix}0\\1\end{bmatrix}\cos\alpha t + \begin{bmatrix}1\\0\end{bmatrix}\sin\alpha t\right\} + A_3(t)\left\{\begin{bmatrix}1\\0\end{bmatrix}\cos\alpha t + \begin{bmatrix}0\\-1\end{bmatrix}\sin\alpha t\right\}\right\}$$
$$= A_1(t)\left\{\begin{bmatrix}\alpha\\0\end{bmatrix} + t\begin{bmatrix}0\\-\alpha\end{bmatrix}\right\} + A_2(t)\left\{\begin{bmatrix}\alpha\\0\end{bmatrix}\cos\alpha t + \begin{bmatrix}0\\-\alpha\end{bmatrix}\sin\alpha t\right\} + A_3(t)\left\{\begin{bmatrix}0\\-\alpha\end{bmatrix}\cos\alpha t + \begin{bmatrix}-\alpha\\0\end{bmatrix}\sin\alpha t\right\}$$

Then

$$\dot{A}_1(t)\left\{\begin{bmatrix}0\\1\end{bmatrix} + t\begin{bmatrix}1\\0\end{bmatrix}\right\} + \dot{A}_2(t)\left\{\begin{bmatrix}0\\1\end{bmatrix}\cos\alpha t + \begin{bmatrix}1\\0\end{bmatrix}\sin\alpha t\right\} + \dot{A}_3(t)\left\{\begin{bmatrix}1\\0\end{bmatrix}\cos\alpha t + \begin{bmatrix}0\\-1\end{bmatrix}\sin\alpha t\right\} + A_1(t)\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$$
$$+ A_2(t)\left\{\begin{bmatrix}\alpha\\0\end{bmatrix}\cos\alpha t + \begin{bmatrix}0\\-\alpha\end{bmatrix}\sin\alpha t\right\} + A_3(t)\left\{\begin{bmatrix}0\\-\alpha\end{bmatrix}\cos\alpha t + \begin{bmatrix}-\alpha\\0\end{bmatrix}\sin\alpha t\right\}$$
$$= A_1(t)\left\{\begin{bmatrix}\alpha\\0\end{bmatrix} + t\begin{bmatrix}0\\-\alpha\end{bmatrix}\right\} + A_2(t)\left\{\begin{bmatrix}\alpha\\0\end{bmatrix}\cos\alpha t + \begin{bmatrix}0\\-\alpha\end{bmatrix}\sin\alpha t\right\} + A_3(t)\left\{\begin{bmatrix}0\\-\alpha\end{bmatrix}\cos\alpha t + \begin{bmatrix}-\alpha\\0\end{bmatrix}\sin\alpha t\right\}$$

$$\dot{A}_1(t)\left\{\begin{bmatrix}0\\1\end{bmatrix} + t\begin{bmatrix}1\\0\end{bmatrix}\right\} + \dot{A}_2(t)\left\{\begin{bmatrix}0\\1\end{bmatrix}\cos\alpha t + \begin{bmatrix}1\\0\end{bmatrix}\sin\alpha t\right\} + \dot{A}_3(t)\left\{\begin{bmatrix}1\\0\end{bmatrix}\cos\alpha t + \begin{bmatrix}0\\-1\end{bmatrix}\sin\alpha t\right\} + A_1 = A_1(t)\left\{\begin{bmatrix}\alpha\\0\end{bmatrix} + t\begin{bmatrix}0\\-\alpha\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix}\right\}$$

Then

$$\begin{bmatrix}t & \sin\alpha t & \cos\alpha t\\1 & \cos\alpha t & -\sin\alpha t\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}\dot{A}_1\\\dot{A}_2\\\dot{A}_3\end{bmatrix} = A_1(t)\begin{bmatrix}\alpha-1\\-\alpha t\end{bmatrix} + \begin{bmatrix}U_X\\U_Y\\0\end{bmatrix} \qquad (14)$$

Equation (14) can be written as

$$t\,\dot{A}_1(t) + \sin\alpha t\,\dot{A}_2(t) + \cos\alpha t\;\dot{A}_3(t) = (\alpha-1)A_1(t) + U_X \qquad (15)$$

$$\dot{A}_1(t) + \cos\alpha t\,\dot{A}_2(t) - \sin\alpha t\;\dot{A}_3(t) = -\alpha\,t\,A_1(t) + U_Y \qquad (16)$$

where $A_1(t)$ is arbitrary function then, put $A_1(t) = 0$ and $\dot{A}_1(t) = 0$, this yields

$$\sin\alpha t\,\dot{A}_2(t) + \cos\alpha t\;\dot{A}_3(t) = U_X \qquad (17)$$

$$\cos\alpha t\,\dot{A}_2(t) - \sin\alpha t\;\dot{A}_3(t) = U_Y \qquad (18)$$

Multiply Equation (17) by $\sin\alpha t$ and Equation (18) by $\cos\alpha t$ and add then

$$(\sin^2\alpha t + \cos^2\alpha t)\;\dot{A}_2(t) = \sin\alpha t\;U_X + \cos\alpha t\;U_Y$$

$$\dot{A}_2(t) = \sin\alpha t\;U_X + \cos\alpha t\;U_Y$$

By integrating we get,

$$A_2(t) = -\frac{U_X}{\alpha}\cos\alpha t + \frac{U_Y}{\alpha}\sin\alpha t \qquad (19)$$

Multiply Equation (17) by $\cos\alpha t$ and Equation (18) by $(-\sin\alpha t\,)$and add then

$$(\sin^2\alpha t + \cos^2\alpha t)\;\dot{A}_3(t) = \cos\alpha t\;U_X - sin\,\alpha t\,U_Y$$

$$\dot{A}_3(t) = \cos\alpha t\quad U_X - \sin\alpha t\,U_Y$$

By integrating we get,

$$A_3(t) = \frac{U_X}{\alpha}\sin\alpha t\ U_x + \frac{U_Y}{\alpha}\cos\alpha t\ U_y \tag{20}$$

Substitute from Equations (19) and (20) into Equation (15) this yields

$$t\dot{A}_1(t) + \sin\alpha t\,(-\frac{U_x}{\alpha}\cos\alpha t + \frac{U_y}{\alpha}\sin\alpha t) + \cos\alpha t\,(\frac{U_x}{\alpha}\sin\alpha t + \frac{U_y}{\alpha}\cos\alpha t) = (\alpha - 1)A_1(t) + U_V$$

$$t\dot{A}_1(t) - (\alpha-1)A_1(t) = U_X - \sin\alpha t\,(-\frac{U_X}{\alpha}\cos\alpha t + \frac{U_Y}{\alpha}\sin\alpha t) - \cos\alpha t\,(\frac{U_X}{\alpha}\sin\alpha t + \frac{U_y}{\alpha}\cos\alpha t)$$

$$t\dot{A}_1(t) - (\alpha-1)A_1(t) = U_X - \frac{U_X}{\alpha}(-\sin\alpha t\cos\alpha t + \sin\alpha t\cos\alpha t) - \frac{U_Y}{\alpha}(\sin^2\alpha t + \cos^2\alpha t)$$

$$t\dot{A}_1(t) - \frac{(\alpha-1)}{t}A_1(t) = \frac{1}{t}(U_X + \frac{U_Y}{\alpha}) \tag{21}$$

Equation (21) represents linear ODE, which will be solved as follows

$$\frac{d}{dt}(A_1(t)\cdot\frac{1}{t^{\alpha-1}}) = \frac{1}{t^{\alpha}}\left(U_X + \frac{U_Y}{\alpha})\right)$$

By integrating to obtain the value of A_1,

$$A_1(t)\cdot\frac{1}{t^{\alpha-1}} = (U_X - \frac{U_Y}{\alpha})\,(-\frac{1}{(\alpha-1)t^{-(\alpha-1)}}) \tag{22}$$

Now, substitute from Equations (19), (20) and (22) into Equation (12) the particular-solution is obtained

$$\chi_p = (U_X - \frac{U_Y}{\alpha})\left(-\frac{1}{(\alpha-1)}\right)\left\{\begin{bmatrix}0\\1\end{bmatrix} + t\begin{bmatrix}1\\0\end{bmatrix}\right\} + (-\frac{U_X}{\alpha}\cos\alpha t + \frac{U_Y}{\alpha}\sin\alpha t)\left\{\begin{bmatrix}0\\1\end{bmatrix}\cos\alpha t + \begin{bmatrix}1\\0\end{bmatrix}\sin\alpha t\right\} + (\frac{U_X}{\alpha}\sin\alpha t + \frac{U_Y}{\alpha}\cos\alpha t)\left\{\begin{bmatrix}1\\0\end{bmatrix}\cos\alpha t + \begin{bmatrix}0\\-1\end{bmatrix}\sin\alpha t\right\} \tag{23}$$

Then the general analytical solutions for the nonhomogeneous system of ODEs (2) is given by

$$\chi(t) = \chi_H + \chi_P \tag{24}$$

$$\chi(t) = \begin{bmatrix}X(t)\\Y(t)\end{bmatrix} = c_1\left\{\begin{bmatrix}0\\1\end{bmatrix} + t\begin{bmatrix}1\\0\end{bmatrix}\right\} + c_2\left\{\begin{bmatrix}0\\1\end{bmatrix}\cos\alpha t + \begin{bmatrix}1\\0\end{bmatrix}\sin\alpha t\right\} + c_3\left\{\begin{bmatrix}1\\0\end{bmatrix}\cos\alpha t + \begin{bmatrix}0\\-1\end{bmatrix}\sin\alpha t +\right\} + (U_X - \frac{U_Y}{\alpha})\left(-\frac{1}{(\alpha-1)}\right)\left\{\begin{bmatrix}0\\1\end{bmatrix} + t\begin{bmatrix}1\\0\end{bmatrix}\right\} + (-\frac{U_X}{\alpha}\cos\alpha t + \frac{U_Y}{\alpha}\sin\alpha t)\left\{\begin{bmatrix}0\\1\end{bmatrix}\cos\alpha t + \begin{bmatrix}1\\0\end{bmatrix}\sin\alpha t\right\} + (\frac{U_X}{\alpha}\sin\alpha t + \frac{U_Y}{\alpha}\cos\alpha t)\left\{\begin{bmatrix}1\\0\end{bmatrix}\cos\alpha t + \begin{bmatrix}0\\-1\end{bmatrix}\sin\alpha t\right\} \tag{25}$$

To apply the Variation of Parameters Method on a dynamical system, the locations of the libration points, Jacobi constant, and the stability of motion will be illustrated as follows.

Locations of the Libration Points

Let $\dot{\boldsymbol{X}} = \dot{\boldsymbol{Y}} = \dot{\boldsymbol{Z}} = \boldsymbol{0}$, and $\ddot{\boldsymbol{X}} = \ddot{\boldsymbol{Y}} = \ddot{\boldsymbol{Z}} = \boldsymbol{0}$, and using Equation (25) at t = 0, to evaluate the initial condition. We put $X(a) = Y(a) = b_1$, and $\dot{X}(a) = \dot{Y}(a) = b_2$.

When $a \to 0$, then, $b_1 \to 10^{-4}$, $b_2 \to 0$. We obtain the values of c_1, c_2 and c_3

Which are given by $c_1 = \frac{\alpha^2 10^{-4} + U_Y}{\alpha(\alpha-1)}$, $c_2 = \frac{\alpha\ 10^{-4} + U_X}{\alpha(\alpha-1)}$ and $c_2 = \frac{\alpha\ 10^{-4} + U_Y}{\alpha(\alpha-1)}$.

These values are used through the system of Equations (2), this yield

$$n^2 X - \frac{(1-\mu)(X-\mu)\,(1-\beta)}{r_1^3} - \frac{\mu(X-\mu+1)}{r_2^3} - \frac{3\,A\,\mu(X-\mu+1)}{2\,r_2^5} = 0 \qquad (26)$$

$$n^2 Y - \frac{(1-\mu)(1-\beta)\,Y}{r_1^3} - \frac{\mu Y}{r_2^3} - \frac{3\,A\,\mu Y}{2\,r_2^5} = 0 \qquad (27)$$

by solving Equation (26) numerically the locations of the collinear libration points are obtained, while the locations of the triangular libration points are calculated by solving Equations (26) and (27) together to obtain (X, Y) for each triangular point.

The Jacobi constant

It is well known that the Jacobi constant is given by Szebehely[19]

$$\mathrm{C} = \left(\dot{\mathrm{X}}^2 + \dot{\mathrm{Y}}^2\right) - 2\mathrm{U} \qquad (28)$$

Since the Jacobi constant can be obtained at each libration point, then this enables to obtain the zero-velocity curves about each point. Since the variation of parameter solution of Equation (25) gives expressions for the (X, and Y) depending on time, which contains trigonometric functions depend on the angular velocity, these terms represent the short periodic orbits around the libration point understudy, the eccentricity and period of the orbit can be obtained by

$$e^2 = \frac{c^2-1}{c^2} \qquad (29)$$

$$T = \frac{2\pi}{S} \qquad (30)$$

where, $c = \frac{\lambda_i^2 - U_{XX}}{2\,\lambda_i - U_{XY}}$, S= Coefficient ofthe imaginary part of λ Ibrahim[20].

Results and Discussion

To apply the variation of parameter solution on the dynamical systems, the Sun-Earth-spacecraft system is considered. A Mathematica Cod is constructed to solve Equations (26) and (27) using Equation (25) to obtain the libration points for this model and to study the motion about each point. Then Table 1 illustrates the locations of libration points, Jacobi constant, eccentricity, and period of the orbit which are obtained also.

The motion about each of the libration points is represented by showing the behavior of the phase space about the libration point understudy, the orbit obtained represents the short periodic orbits and it is obtained from Equation (25) with $c_1 = 0$. Zero velocity curves show the regions at which the infinitesimal body moves with different values of energy levels. The first level is at the initial values of the Jacobi constant. Finally, the Poincare surfaces of section is a technique which represented the stability of the motion about the libration points, it is a projection of the orbits from $(X, Y, \dot{X})$ plane to $(X, \dot{X})$ plane, each point represents an orbit about the libration point.

Now, L1and L2 are chosen as an example of the collinear libration points, while L4 and L6 are chosen as an example of the triangular libration points.

From the results obtained Fig 2 shows periodic orbit about L1, Fig. 3 illustrates the zero-velocity curve and Fig.4 shows a Poincare surface about L1 and Fig 5 shows periodic orbit about L2, Fig. 6 illustrates the zero-velocity curve and Fig.7 shows a Poincare surface about L2. The same for L4, Fig 8 shows the periodic orbit about L4, Fig. 9 shows the zero-velocity curves and Fig.10 Poincare surface about L4. The same for L6 is illustrated in figures 11, 12, and 13.

Table 1: The collinear and non-collinear libration points for the Sun-Earth system and their Jacobi constant C, eccentricity and period of the orbit about each point.

libration points	Position		Jacobi constant	ccentricity	period of the orbit
	X	Y			
L1	0.986675	0	2.98435	0.978755	3.13455
L2	1.01201	0	2.98488	0.991976	3.13455
L3	-1.00332	0	2.98463	1.6196	3.13455
L4	0.986675	0.997497	3.3886	0.856303	3.13455
L5	0.986675	-1.0025	3.39508	0.451298	3.13455
L6	0.986675	9.00001*10^-10	2.98435	0.978755	3.13455

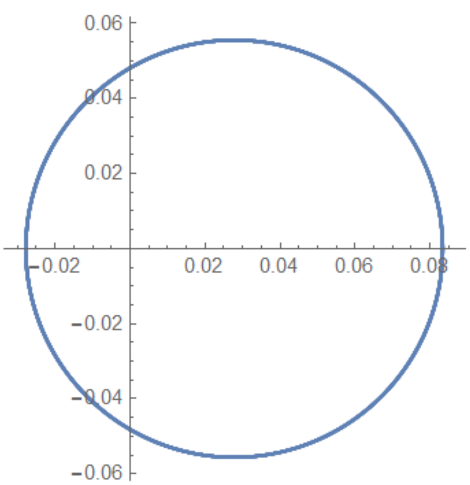


Fig.2: the phase space about L1

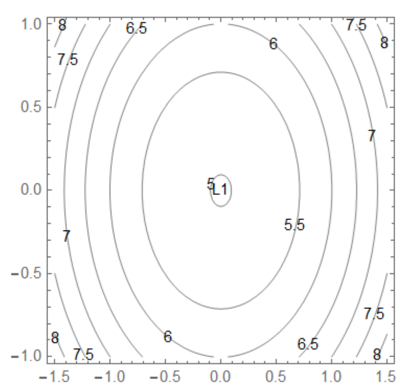


Fig. 3. Zero Velocity Curve at L1 , C=2.98435

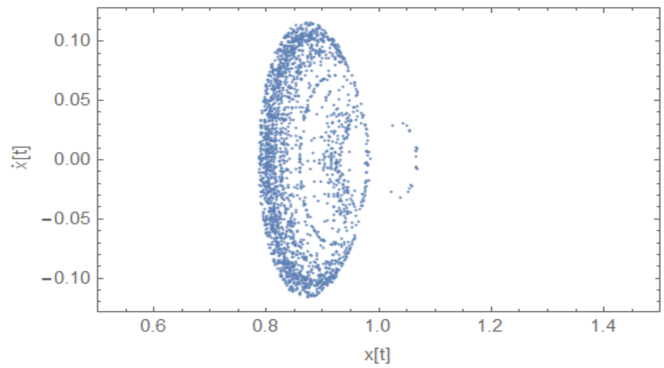
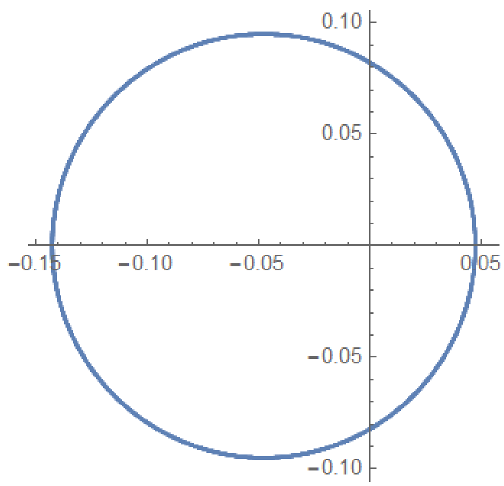


Fig. 4. Poincare Surface at L1



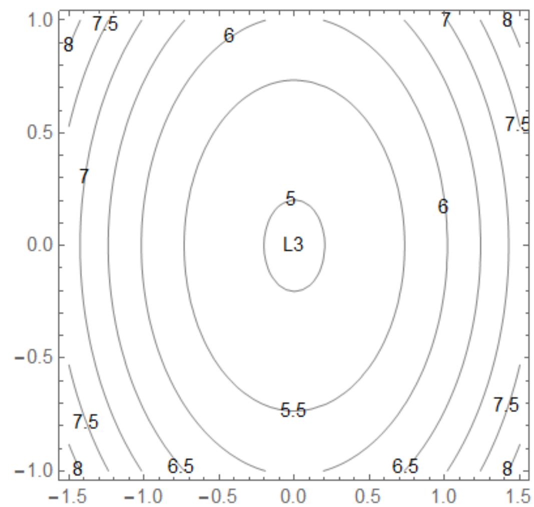


Fig.5: the phase space about L2

Fig. 6. Zero Velocity Curve at L2 , C=2.98488

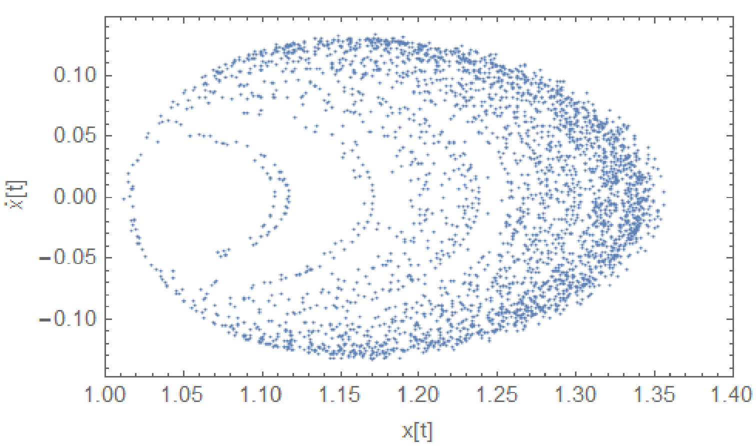


Fig. 7. - Poincare Surface at L2

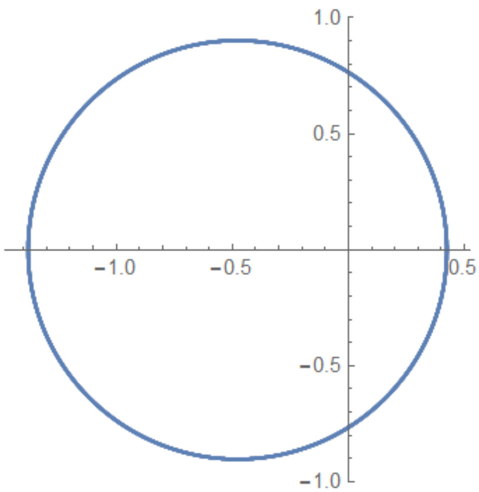


Fig. 8:Phase Space at L4

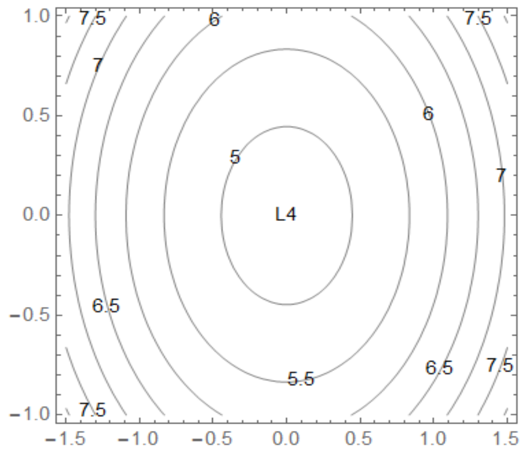


Fig. 9:Zero Velocity Curve at L 4 ,C = 3.3886

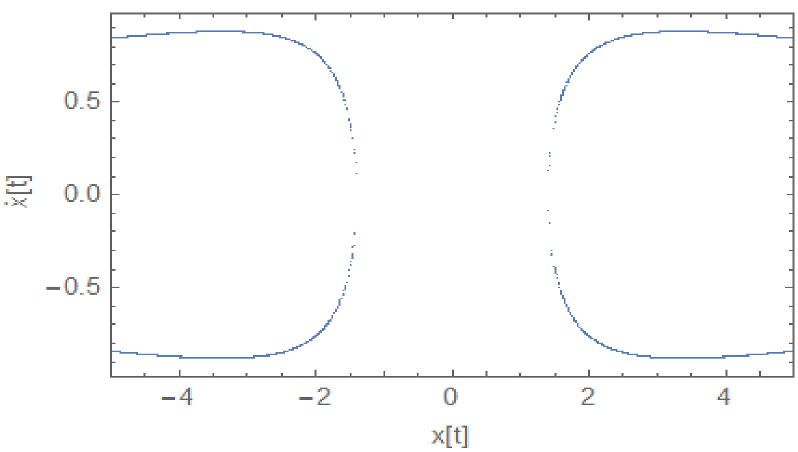
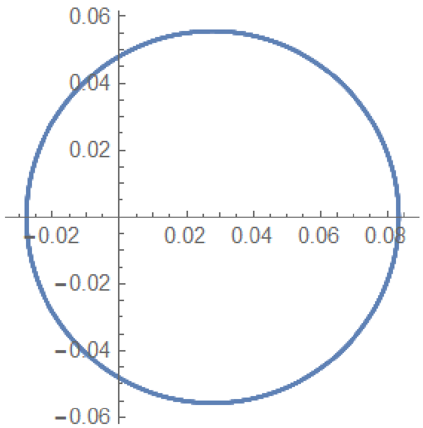


Fig. 10: Poincare Surface at L4



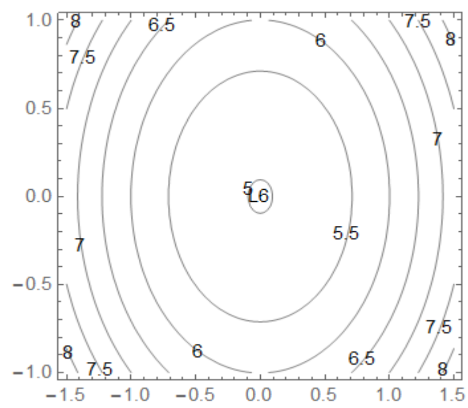


Fig. 11: Phase Space at L6 **Fig.12:Zero Velocity Curve at L6 ,C =** 2.98435

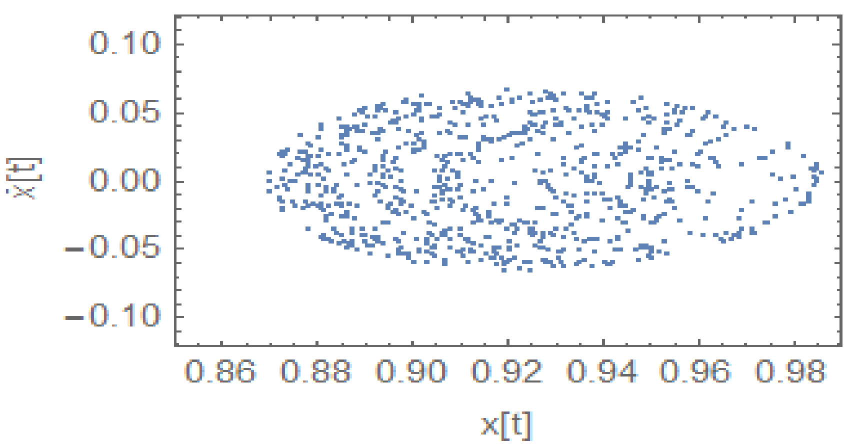


Fig. 13: Poincare Surface at L6

Conclusion

In this study, the Variation of Parameters Method is used to obtain the analytical solution of the restricted three-body problem. The obtained solution gives explicit expressions in X, Y depends on the time. The application of these solutions enables us to obtain the libration points and to study the stability of motion about these libration points, the results obtained are in a good agreement with the results obtained by Simmons [13], Kunitsyn[14], Schuerman[15] and Ibrahim[16]. Therefore the variation of the parameter method is a good technique to be applied on the astro-dynamical systems.

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