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FIXED POINT RESULTS FOR RATIONAL CONTRACTIONS INVOLVING CONTROL FUNCTIONS IN COMPLEX VALUED B-METRIC SPACES

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Abstract. In this paper, we prove common fixed point theorems for a pair of mappings with rational contractions having control functions as a coefficients in complex valued b-metric spaces. Our results generalize and extend some known results in the literature.

1. Introduction

The concept of complex valued b-metric space was introduced by Rao et. al.[11], which was more general than the complex valued metric spaces[1]. They proved some fixed point results for mappings satisfying a rational inequality in complex valued b-metric spaces. Since then, several paper have dealt with fixed point theorems in complex valued b-metric spaces (see [2-10],[12] and references therein).

The aim of this paper is to consider and establish results on the setting of complex valued b-metric spaces, regarding common fixed points of two mappings, using a rational contractions invovling control functions.

2. Preliminaries

We recall some notations and definitions that will be needed in the sequel.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$.Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$**:** if and only if $Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2)$.

Consequently, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied :

(i) $Re(z_1) = Re(z_2), Im(z_1) < Im(z_2)$;**:**

(ii) $Re(z_1) < Re(z_2), Im(z_1) = Im(z_2)$;**:**

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(iii) $Re(z_1) < Re(z_2), Im(z_1) < Im(z_2)$;:
(iv) $Re(z_1) = Re(z_2), Im(z_1) = Im(z_2)$.:

In particular, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i),(ii) and (iii) is satisfied and we write $z_1 \prec z_2$ if only (iii) is satisfied. Notice that

(a): if $0 \precsim z_1 \precnsim z_2$, then $|z_1| < |z_2|$;
(b): if $z_1 \precsim z_2$ and $z_2 \prec z_3$ then $z_1 \prec z_3$;
(c): if $a, b \in \mathbb{R}$ and $a \leq b$ then $az \precsim bz$ for all $z \in \mathbb{C}_+$.

The: following definition is recently introduced by Rao et al. [11].

Definition 2.1. **[11]** Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(i): $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$.
(ii): $d(x, y) = d(y, x)$.
(iii): $d(x, y) \precsim s[d(x, z) + d(z, y)]$.

The: pair (X, d) is called a complex valued b-metric space.

Example 2.2.[11] If $X = [0, 1]$, define a mapping $d : X \times X \to \mathbb{C}$ by $d(x, y) = |x - y|^2 + i|x - y|^2$, for all $x, y \in X$. Then (X, d) is complex valued b-metric space with $s = 2$.

Definition 2.3.[11] Let (X, d) be a complex valued b-metric space.

(i) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A$.

(ii) A point $x \in X$ is called a limit point of a set A whenever for every $0 \prec r \in \mathbb{C}, B(x, r) \cap (A - \{x\}) \neq \phi$.

(iii) A subset $A \subseteq X$ is called an open set whenever each element of A is an interior point of A.

(iv) A subset $A \subseteq X$ is called closed set whenever each limit point of A belongs to A.

(v) The family $F = \{B(x, r) : x \in X \, and \, 0 \prec r\}$ is a sub-basis for a Hausdorff topology τ on X.

Definition 2.4.[11] Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X and $x \in X$.

(i) If for every $c \in \mathbb{C}$, with $0 \prec c$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent and converges to x. We denote this by $lim_{n\to\infty} x_n = x$ or $\{x_n\} \to x \, as \, n \to \infty$.

(ii) If for every $c \in \mathbb{C}$, with $0 \prec c$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$is said to be a Cauchy sequence.

(iii) If every Cauchy sequence in X is convergent in X, then (X, d) is said to be a complete complex valued b-metric space.

Lemma 2.5.[11] Let (X,d) be a complex valued b-metric space and let $\{x_n\}$be a sequence in X. Then $\{x_n\}$converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.

Lemma 2.6.[11] Let (X,d) be a complex valued b-metric space and let $\{x_n\}$be a sequence in X. Then $\{x_n\}$ is Cauchy sequence if and only if $|d(x_n,x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.

3. Main Result

Theorem 3.1. Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S,T : X \to X$ be mappings. If there exist mappings $\alpha,\beta,\gamma,\delta : X \to [0,1)$ such that for all $x,y \in X$:

(i) $\propto (Sx) \leq \alpha(x), \beta(Sx) \leq \beta(x), \gamma(Sx) \leq \gamma(x)\, and\, \delta(Sx) \leq \delta(x)$;

(ii)$\propto (Tx) \leq \alpha(x), \beta(Tx) \leq \beta(x), \gamma(Tx) \leq \gamma(x)\, and\, \delta(Tx) \leq \delta(x)$;:

(iii)$\propto (x) + \beta(x) + 2\gamma(x) \leq 2s\delta(x) < 1$;:

(iv)
$$d(Sx,Ty) \precsim \alpha(x)d(x,y) + \frac{\beta(x)d(y,Ty)d(x,Sx)}{1+d(x,y)} + \gamma(x)[d(x,Sx)+d(y,Ty)] + \delta(x)[d(x,Ty)+d(y,Sx)]. \quad (3.1)$$

Then S and T have a unique common fixed point.

Proof. For any arbitrary point $x_0 \in X$. Since $S(X) \subseteq X\, and\, T(X) \subseteq X$, we can define sequence $\{x_n\}$ in X such that

$$x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1}, \text{ for } n \geq 0. \quad (3.2)$$

Now, we show that the seuqence $\{x_n\}$ is Cauchy. Let $x = x_{2n}\, and\, y = x_{2n+1}$ in (3.1), we get

$$\begin{aligned} d(Sx_{2n},Tx_{2n+1}) &= d(x_{2n+1},x_{2n+2}) \\ &\precsim \alpha(x_{2n})d(x_{2n},x_{2n+1}) + \frac{\beta(x_{2n})d(x_{2n+1},Tx_{2n+1})d((x_{2n},Sx_{2n})}{1+d((x_{2n},x_{2n+1})} \\ &\quad +\gamma(x_{2n})[d(x_{2n},Sx_{2n})+d(x_{2n+1},Tx_{2n+1})] \\ &\quad +\delta(x_{2n})[d(x_{2n},Tx_{2n+1})+d(x_{2n+1},Sx_{2n})] \\ &= \alpha(x_{2n})d(x_{2n},x_{2n+1}) + \frac{\beta(x_{2n})d(x_{2n+1},x_{2n+2})d((x_{2n},x_{2n+1})}{1+d((x_{2n},x_{2n+1})} \\ &\quad +\gamma(x_{2n})[d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})] \\ &\quad +\delta(x_{2n})[d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+1})] \end{aligned}$$

which implies that

$$\begin{aligned} |d(x_{2n+1},x_{2n+2})| &\leq \alpha(x_{2n})|d(x_{2n},x_{2n+1})| \\ &\quad + \frac{\beta(x_{2n})|d(x_{2n+1},x_{2n+2})||d((x_{2n},x_{2n+1})|}{|1+d((x_{2n},x_{2n+1})|} \\ &\quad +\gamma(x_{2n})|d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})| \\ &\quad +s\delta(x_{2n})|d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})|. \end{aligned}$$

Since $|1+d(x_{2n},x_{2n+1})| \geq |d(x_{2n},x_{2n+1})|$,

$$\begin{aligned} |d(x_{2n+1},x_{2n+2})| &\leq \alpha(x_{2n})|d(x_{2n},x_{2n+1})| \\ &\quad +\beta(x_{2n})|d(x_{2n+1},x_{2n+2})| \\ &\quad +\gamma(x_{2n})|d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})| \\ &\quad +s\delta(x_{2n})|d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})| \end{aligned}$$

and so

$$= \alpha(Tx_{2n-1})|d(x_{2n}, x_{2n+1})| + \beta(Tx_{2n-1})|d(x_{2n+1}, x_{2n+2})|:$$
$$+\gamma(Tx_{2n-1})|d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})|:$$
$$+s\delta(Tx_{2n-1})|d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})|:$$
$$|d(x_{2n+1}, x_{2n+2})| \leq \alpha(x_{2n-1})|d(x_{2n}, x_{2n+1})|:$$
$$+\beta(x_{2n-1})|d(x_{2n+1}, x_{2n+2})|:$$
$$+\gamma(x_{2n-1})|d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})|:$$
$$+s\delta(x_{2n-1})|d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})|:$$
$$= \alpha(Sx_{2n-2})|d(x_{2n}, x_{2n+1})| + \beta(Sx_{2n-2})|d(x_{2n+1}, x_{2n+2})|:$$
$$+\gamma(Sx_{2n-2})|d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})|:$$
$$+s\delta(Sx_{2n-2})|d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})|:$$
$$\leq \alpha(x_{2n-2})|d(x_{2n}, x_{2n+1})| + \beta(x_{2n-2})|d(x_{2n+1}, x_{2n+2})|:$$
$$+\gamma(x_{2n-2})|d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})|:$$
$$+s\delta(x_{2n-2})|d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})|:$$
$$-----------:$$
$$-----------:$$
$$\leq \alpha(x_0)|d(x_{2n}, x_{2n+1})| + \beta(x_0)|d(x_{2n+1}, x_{2n+2})|:$$
$$+\gamma(x_0)|d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})|:$$
$$+s\delta(x_0)|d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})|:$$

which yields that

$$|d(x_{2n+1}, x_{2n+2})| \leq \frac{\propto(x_0)+\gamma(x_0)+s\delta(x_0)}{1-\beta(x_0)-\gamma(x_0)-s\delta(x_0)}|d(x_{2n}, x_{2n+1})|. \qquad (3.3)$$

Similarly, one can obtain

$$|d(x_{2n+2}, x_{2n+3})| \leq \frac{\propto(x_0)+\gamma(x_0)+s\delta(x_0)}{1-\beta(x_0)-\gamma(x_0)-s\delta(x_0)}|d(x_{2n+1}, x_{2n+2})|. \qquad (3.4)$$

Let $\mu = \frac{\propto(x_0)+\gamma(x_0)+s\delta(x_0)}{1-\beta(x_0)-\gamma(x_0)-s\delta(x_0)} < 1$.

Since $\propto(x_0) + \beta(x_0) + 2\gamma(x_0) + 2s\delta(x_0) < 1$, thus we have

$|d(x_{2n+1}, x_{2n+2})| \leq \mu|d(x_{2n}, x_{2n+1})|$ and

$|d(x_{2n+2}, x_{2n+3})| \leq \mu|d(x_{2n+1}, x_{2n+2})|$, or in fact

$$|d(x_{n+1}, x_{n+2})| \leq \mu|d(x_n, x_{n+1})|. \qquad (3.5)$$

or $|d(x_n, x_{n+1})| \leq \mu^n|d(x_0, x_1)|$. (3.6)

Thus for any $m > n, m, n \in \mathbb{N}$ and since $s\mu < 1$, we get

$$|d(x_n, x_m)| \leq s|d(x_n, x_{n+1})| + s|d(x_{n+1}, x_m)|$$
$$\leq s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| + s^2|d(x_{n+2}, x_m)|,$$

continuing in the manner, we get

$$|d(x_n, x_m)| \leq s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| + s^3|d(x_{n+2}, x_{n+3})| - - - +$$
$$- - - - +s^{m-n-1}|d(x_{m-2}, x_{m-1})| + s^{m-n}|d(x_{m-1}, x_m)|.$$

By using (3.6), we get

$$|d(x_n,x_m)| \leq s\mu^n|d(x_0,x_1)| + s^2\mu^{n+1}|d(x_0,x_1)| + --- + s^{m-n}\mu^{m-1}|d(x_0,x_1)|$$
$$= \sum_{i=1}^{m-n} s^i\mu^{i+n-1}|d(x_0,x_1)|.$$

Therefore,

$$|d(x_n,x_m)| \leq \sum_{i=1}^{m-n} s^{i+n-1}\mu^{i+n-1}|d(x_0,x_1)|$$
$$= \sum_{t=n}^{m-1} s^t\mu^t|d(x_0,x_1)|$$
$$\leq \sum_{t=n}^{\infty}(s\mu)^t|d(x_0,x_1)| = \tfrac{(s\mu)^n}{1-s\mu}|d(x_0,x_1)|. \qquad (3.7):$$

Therefore $|d(x_n,x_m)| \leq \frac{(s\mu)^n}{1-s\mu}|d(x_0,x_1)| \to 0 \, as \, m,n \to \infty$.

Thus $\{x_n\}$is a Cauchy sequence in X. By completeness of X, there exists a point $u \in X$ such that $x_n \to u$ as $n \to \infty$.

Next we claim that $Su = u$.

Assume not, then there exists $z \in X$ such that

$|d(u,Su)| = |z| > 0. \qquad (3.8)$

So by using the notion of a complex valued b-metric, we have

$$z = d(u,Su) \precsim sd(u,x_{2n+2}) + sd(x_{2n+2},Su)$$
$$= sd(u,x_{2n+2}) + sd(Su,Tx_{2n+1})$$
$$\precsim sd(u,x_{2n+2}) + s\propto(u)d(u,x_{2n+1})$$
$$+\tfrac{s\beta(u)d(x_{2n+1},Tx_{2n+1})d(u,Su)}{1+d(u,x_{2n+1})}$$
$$+s\gamma(u)[d(u,Su) + d(x_{2n+1},Tx_{2n+1})]$$
$$+s\delta(u)[d(u,Tx_{2n+1}) + d(x_{2n+1},Su)]$$
$$= sd(u,x_{2n+2}) + s\propto(u)d(u,x_{2n+1})$$
$$+\tfrac{s\beta(u)d(x_{2n+1},x_{2n+2})d(u,Su)}{1+d(u,x_{2n+1})}$$
$$+s\gamma(u)[d(u,Su) + d(x_{2n+1},x_{2n+2})]$$
$$+s\delta(u)[d(u,x_{2n+2}) + d(x_{2n+1},Su)]$$

which implies that

$$|z| = |d(u,Su)| \leq s|d(u,x_{2n+2})| + s\propto(u)|d(u,x_{2n+1})|$$
$$+\tfrac{s\beta(u)|d(x_{2n+1},x_{2n+2})||d(u,Su)|}{|1+d(u,x_{2n+1})|}$$
$$+s\gamma(u)|d(u,Su) + d(x_{2n+1},x_{2n+2})|$$
$$+s\delta(u)|d(u,x_{2n+2}) + d(x_{2n+1},Su)|. \qquad (3.9)$$

Taking the limit of (3.9) as $n \to \infty$, we get that

$$|z| = |d(u,Su)| \leq s\gamma(u)|d(u,Su)| + s\delta(u)|d(u,Su)| = s[\gamma(u)+\delta(u)]|d(u,Su)|$$

$\leq s[\propto(u)+\beta(u)+2\gamma(u)+2\delta(u)]|d(u,Su)|$

$< |d(u,Su)|,:$

a contradiction and so $|d(u,Su)|=0$; that is $u=Su$. It follows similarly that $u=Tu$. This implies that u is a common fixed point of S and T.

We now prove that this u is unique.

$$d(u,u^\star)=d(Su,Tu^\star)$$
$$\precsim \propto(u)d(u,u^\star)+\frac{\beta(u)d(u^\star,Tu^\star)d(u,Su)}{1+d(u,u^\star)}+\gamma(u)[d(u,Su)+d(u^\star,Tu^\star)]$$
$$+\delta(u)[d(u,Tu^\star)+d(u^\star,Su)]$$
$$\precsim \propto(u)d(u,u^*)+2\delta(u)d(u,u^\star).$$

Therefore, we have

$$|d(u,u^\star)|\leq[\propto(u)+2\delta(u)]|d(u,u^\star)|. \qquad (3.10)$$

Since $\propto(u)+2\delta(u)<1$, we have $|d(u,u^\star)|=0$.

Thus $u=u^\star$, which proves the uniqueness of common fixed point in X. This concludes the theorem.

Theorem 3.2. Let (X,d) be a complete complex valued b-metric space with the coefficient $s\geq 1$ and let $T:X\to X$ be a mapping. If there exist mappings $\propto,\beta,\gamma,\delta:X\to[0,1)$ such that for all $x,y\in X$:

(i) $\propto(Tx)\leq\alpha(x),\beta(Tx)\leq\beta(x),\gamma(Tx)\leq\gamma(x)\,and\,\delta(Tx)\leq\delta(x)$;

(ii) $\propto(x)+\beta(x)+2\gamma(x)+2s\delta(x)<1$;

(iii) $$d(Tx,Ty)\precsim\propto(x)d(x,y)+\frac{\beta(x)[1+d(x,Tx)]d(y,Ty)}{1+d(x,y)}:$$
$$+\gamma(x)[d(x,Tx)+d(y,Ty)]:$$
$$+\delta(x)[d(x,Ty)+d(y,Tx)]. \qquad (3.11):$$

Then T has a unique fixed point.

Proof.: Let $x_0\in X$ and the sequence $\{x_n\}$ be defined by $x_{n+1}=Tx_n$, where $n=0,1,2,---$. (3.12)

Now: we show that $\{x_n\}$ is a Cauchy sequence. From condition (3.11), we have

$$d(x_{n+1},x_{n+2})=d(Tx_n,Tx_{n+1}):$$
$$\precsim\propto(x_n)d(x_n,x_{n+1})+\frac{\beta(x_n)[1+d(x_n,Tx_n)]d(x_{n+1},Tx_{n+1})}{1+d(x_n,x_{n+1})}:$$
$$+\gamma(x_n)[d(x_n,Tx_n)+d(x_{n+1},Tx_{n+1})]:$$
$$+\delta(x_n)[d(x_n,Tx_{n+1})+d(x_{n+1},Tx_n)]:$$
$$=\propto(x_n)d(x_n,x_{n+1})+\frac{\beta(x_n)[1+d(x_n,x_{n+1})]d(x_{n+1},x_{n+2})}{1+d(x_n,x_{n+1})}:$$
$$+\gamma(x_n)[d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})]:$$
$$+\delta(x_n)[d(x_n,x_{n+2})+d(x_{n+1},x_{n+1})]:$$

which implies that

$$|d(x_{n+1},x_{n+2})|\leq\propto(x_0)|d(x_n,x_{n+1})|+\beta(x_0)|d(x_{n+1},x_{n+2})|:$$
$$+\gamma(x_0)|d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})|:$$

$$+s\delta(x_0)|d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})|:$$

which yields that

$$|d(x_{n+1},x_{n+2})| \leq \frac{\propto(x_0)+\gamma(x_0)+s\delta(x_0)}{1-\beta(x_0)-\gamma(x_0)-s\delta(x_0)}|d(x_n,x_{n+1})|. \qquad (3.13):$$

Similarly, one can obtain

$$|d(x_{n+2},x_{n+3})| \leq \frac{\propto(x_0)+\gamma(x_0)+s\delta(x_0)}{1-\beta(x_0)-\gamma(x_0)-s\delta(x_0)}|d(x_{n+1},x_{n+2})|. \qquad (3.14):$$

Let $\mu = \frac{\propto(x_0)+\gamma(x_0)+s\delta(x_0)}{1-\beta(x_0)-\gamma(x_0)-s\delta(x_0)} < 1$,

Since $\alpha(x_0)+\beta(x_0)+2\gamma(x_0)+2s\delta(x_0) < 1$, thus we have

$$|d(x_n,x_{n+1})| \leq \mu^n|d(x_0,x_1)|. \qquad (3.15):$$

By the same line of action as in the previous Theorem 3.1, we have $\{x_n\}$ is a Cauchy sequencee in X. Since X is complete, there exists some $u \in X$ such that $x_n \to u$ as $n \to \infty$. Next we show that u is a fixed point of T.

From (3.11), we have $d(u,Tu) \precsim sd(u,Tx_n)+sd(Tx_n,Tu)$

$$\precsim sd(u,Tx_n)+s\propto(x_n)d(x_n,u)+\frac{+s\beta(x_n)[1+d(x_n,Tx_n)]d(u,Tu)}{1+d(x_n,u)}$$
$$+s\gamma(x_n)[d(x_n,Tx_n)+d(u,Tu)]$$
$$+s\delta(x_n)[d(x_n,Tu)+d(u,Tx_n)].$$

This implies that

$$|d(u,Tu)| \leq s|d(u,x_{n+1})|+s\alpha(x_0)|d(x_n,u)|$$
$$+\frac{s\beta(x_0)|1+d(x_n,x_{n+1})||d(u,Tu)|}{|1+d(x_n,u)|}$$
$$+s\gamma(x_0)|d(x_n,x_{n+1})+d(u,Tu)|$$
$$+s\delta(x_0)|d(x_n,Tu)+d(u,x_{n+1})|$$

which on making $n \to \infty$ reduces to

$$|d(u,Tu)| \leq s\beta(x_0)|d(u,Tu)|+s\gamma(x_0)|d(u,Tu)|+s\delta(x_0)|d(u,Tu)|$$
$$=[s\beta(x_0)+s\gamma(x_0)+s\delta(x_0)]|d(u,Tu)|$$
$$\leq s[\alpha(x_0)+\beta(x_0)+2\gamma(x_0)+2\delta(x_0)]|d(u,Tu)|, \qquad (3.16)$$

a contradiction, and so $|d(u,Tu)| = 0$; that is, $u = Tu$. This implies that u is a fixed point of T.

Uniqueness of fixed point is an easy consequence of condition (3.11). This completes the proof.

Corollary 3.3. Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mapping. If there exist mappings $\propto,\beta,\gamma,\delta : X \to [0,1)$ such that for all $x,y \in X$ and for some fixed n :

(i) $\propto(T^nx) \leq \alpha(x), \beta(T^nx) \leq \beta(x), \gamma(T^nx) \leq \gamma(x)$ and $\delta(T^nx) \leq \delta(x)$;

(ii)$\alpha(x)+\beta(x)+2\gamma(x)+2s\delta(x) < 1$;:

(iii)$d(T^nx,T^ny) \precsim \alpha(x)d(x,y)+\frac{\beta(x)[1+d(x,T^nx)]d(y,T^ny)}{1+d(x,y)}$:

$$+\gamma(x)[d(x,T^nx)+d(y,T^ny)]:$$

$$+\delta(x)[d(x,T^ny)+d(y,T^nx)]. \qquad (3.17):$$

Then T has a unique fixed point.

Proof.: By Theorem 3.2 there exists $v \in X$ such that $T^nv = v$. Then

$d(Tv,v) = d(TT^nv,T^nv) = d(T^nTv,T^nv)$:

$\precsim \propto (Tv)d(Tv,v) + \frac{\beta(Tv)[1+d(Tv,T^nTv)]d(v,T^nv)}{1+d(Tv.v)}$:

$+\gamma(Tv)[d(Tv,T^nTv) + d(v,T^nv)]$:

$+\delta(Tv)[d(Tv,T^nv) + d(v,T^nTv)]$:

$= \propto (Tv)d(Tv,v) + \frac{\beta(Tv)[1+d(Tv,T^nTv)]d(v,v)}{1+d(Tv.v)}$:

$+\gamma(Tv)[d(Tv,T^nTv) + d(v,v)]$:

$+\delta(Tv)[d(Tv,v) + d(v,T^nTv)]$:

$\precsim \propto (Tv)d(Tv,v) + 2\delta(Tv)d(Tv,v)$:

$= (\propto +2\delta)(Tv)d(Tv,v)$:

and so $d(Tv,v) = 0$. So $Tv = v$. Therefore, the fixed point of T is unique.

References

[1] A. Azam, B. Fisher, M. Khan, Common fixed point theorems in complex valued metric spaces, Numer. Funct. Anal. Optim. 32 (3) (2011), 243-253.

[2] A.K. Dubey, Common fixed point results for contractive mappings in complex valued b-metric spaces, Nonlinear Funct. Anal. Appl. 20 (2) (2015), 257-268.

[3] A.K. Dubey, Complex Valued b-Metric Spaces and Common Fixed Point Theorems under Rational Contractions, J. Complex Anal. 2016 (2016), Article ID 9786063, 7 Pages.

[4] A.K. Dubey and Manjula Tripathi, Some generalized common fixed point theorems under rational contractions in Complex valued b-metric spaces and Applications, Asian J. Math. Appl. 2016 (2016), Article ID ama0323, 10 pages.

[5] Anil Kumar Dubey and Manjula Tripathi, Common Fixed Point Theorem in Complex Valued b-Metric Space for Rational Contractions, Journal of Informatics and Mathematical Sciences, Vol.7, No.3(2015), pp.149-161.

[6] Anil Kumar Dubey, Manjula Tripathi and Ravi Prakash Dubey, Various Fixed Point Theorems in Complex Valued b-Metric Spaces, Int. J. Eng. Math. 2016 (2016), Article ID 7072606, 7 Pages.

[7] A.A. Mukheimer, Some common fixed point theorems in complex valued b-metric spaces, Sci. World J. 2014 (2014), Article ID 587825, 6 pages.

[8] A.A. Mukheimer, Common fixed point theorems for a pair of mappings in complex valued b-metric spaces, Adv. Fixed Point Theory, 4 (3) (2014), 344-354.

[9] Deepak Singh, O.P. Chauhan, Naval Singh, Vishal Joshi, Common fixed point theorems in complex valued b-metric spaces, Adv. Fixed Point Theory, 5 (2) (2015), 263-280.

[10] Ebrahim Analouei Adegani, Ahmad Zireh and Mahmood Bidkham, Common Fixed Point Theorem for Contractive Mappings in Complex Valued b-Metric Spaces, Gazi Univ. J. Sci. (GUJ Sci) 29 (3) (2016), 681-693.

[11] K.P.R Rao, P.R Swamy, J.R., Prasad, A common fixed point theorem in complex valued b-metric spaces, Bull. Math. Stat. Res. 1 (1) (2013), 1-8.

[12] Sunanda R. Patil, J.N. Salunke, Common fixed point theorems for weakly compatible maps in complex valued b-metric spaces, Adv. Inequal, Appl. 2016 (2016), No. 8, 1-20.