INTEGRAL TRANSFORMS OF THE HARMONIC SAWTOOTH MAP, THE RIEMANN ZETA FUNCTION, FRACTAL STRINGS, AND A FINITE REFLECTION FORMULA

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ABSTRACT. The harmonic sawtooth map $w(x)$ of the unit interval onto itself is defined. It is shown that its fixed points $\{x : w(x) = x\}$ are enumerated by the n-th derivatives of a Meijer-G function and Lerch transcendent, serving as exponential and ordinary generating functions respectively, and involving the golden ratio in their parameters. The appropriately scaled Mellin transform of $w(x)$ is an analytic continuation of the Riemann zeta function $\zeta(s)$ valid $\forall -\operatorname{Re}(s) \notin \mathbb{N}$. The series expansion of the inverse scaling function associated to the Mellin transform of $w(x)$ has coefficients enumerating the Large Schröder Numbers S_n, defined as the number of perfect matchings in a triangular grid of n squares and expressible as a hypergeometric function. A finite-sum approximation to $\zeta(s)$ denoted by $\zeta_w(N; s)$ is examined and an associated function $\chi(N; s)$ is found which solves the reflection formula $\zeta_w(N; 1-s) = \chi(N; s)\,\zeta_w(N; s)$. The function $\chi(N; s)$ is singular at $s = 0$ and the residue at this point changes sign from negative to positive between the values of $N = 176$ and $N = 177$. Some rather elegant graphs of the reflection functions $\chi(N; s)$ are also provided. The Mellin and Laplace transforms of the individual component functions of the infinite sums and their roots are compared. The Gauss map $h(x)$ is recalled so that its fixed points and Mellin transform can be contrasted to those of $w(x)$. The geometric counting function $N_{\mathcal{L}_w}(x) = \left\lfloor \frac{\sqrt{2x+1}}{2} - \frac{1}{2} \right\rfloor$ of the fractal string $\mathcal{L}_w$ associated to the lengths of the harmonic sawtooth map components $\{w_n(x)\}_{n=1}^{\infty}$ happens to coincide with the counting function for the number of Pythagorean triangles of the form $\{(a, b, b+1) : (b+1) \leqslant x\}$. The volume of the inner tubular neighborhood of the boundary of the map $\partial \mathcal{L}_w$ with radius ε is shown to have the particularly simple closed-form $V_{\mathcal{L}_w}(\varepsilon) = \frac{4\varepsilon v(\varepsilon)^2 - 4\varepsilon v(\varepsilon) + 1}{2v(\varepsilon)}$ where $v(\varepsilon) = \left\lfloor \frac{\varepsilon + \sqrt{\varepsilon^2 + \varepsilon}}{2\varepsilon} \right\rfloor$. Also, the Minkowski content of $\mathcal{L}_w$ is shown to be $\mathcal{M}_{\mathcal{L}_w} = 2$ and the Minkowski dimension to be $D_{\mathcal{L}_w} = \frac{1}{2}$ and thus not invertible. The geometric zeta function, which is the Mellin transform of the geometric counting function $N_{\mathcal{L}_w}(x)$, is calculated and shown to have a rather unusual closed form involving a finite sum of Riemann zeta functions and binomioal coefficients. Some definitions from the theory of fractal strings and membranes are also recalled.

CONTENTS

1. Unit Interval Mappings

1.1. The n-th Harmonic Sawtooth Function $w_n(x)$.

1.1.1. *Infinite Sum Decomposition.* Let the n-th component $w_n(x) \in [0,1] \forall x \in [0,1]$ of the harmonic sawtooth function $w(x)$ [9, 2.3] be defined as

$$(1) \qquad\qquad w_n(x) \quad = n(xn + x - 1)\chi\left(x, I_n^H\right)$$

where

$$(2) \qquad\qquad \chi\left(x, I_n^H\right) = \begin{cases} 1 & \dfrac{1}{n+1} < x \leqslant \dfrac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

is the characteristic function of the n-th harmonic interval(132). By setting $n = \lfloor x^{-1} \rfloor$ as in (134) we get the unit interval mapping

$$(3) \qquad \begin{aligned} w(x) \quad &= w_{\lfloor x^{-1} \rfloor}(x) \\ &= \lfloor x^{-1} \rfloor \left(x \lfloor x^{-1} \rfloor + x - 1\right)\chi\left(x, I_{\lfloor x^{-1} \rfloor}^H\right) \\ &= \sum_{n=1}^{\infty} w_n(x) \\ &= \sum_{n=1}^{\infty} n(xn + x - 1)\chi\left(x, I_n^H\right) \\ &= \lfloor x^{-1} \rfloor \left(x \lfloor x^{-1} \rfloor + x - 1\right) \\ &= x \lfloor x^{-1} \rfloor^2 + x \lfloor x^{-1} \rfloor - \lfloor x^{-1} \rfloor \end{aligned}$$

As can be seen in Figure 1, $w(x)$ is discontinuous at a countably infinite set of points of Lebesgue measure zero

$$(4) \qquad \begin{aligned} \mathbb{H} \quad &= \left\{ y : \lim_{x \to y^-} w(x) \neq \lim_{x \to y^+} w(x) \right\} \\ &= \left\{ 0, \tfrac{1}{n} : n \in \mathbb{Z} \right\} \end{aligned}$$

The left and right limits at the discontinuous points are

$$(5) \qquad \begin{aligned} \lim_{x \to \in \mathbb{H}^-} w(x) \quad &= 1 \\ \lim_{x \to \in \mathbb{H}^+} w(x) \quad &= 0 \end{aligned}$$

1.1.2. *The Fixed Points of $w(x)$ as an Iterated Function System.* The iterates of the map

$$(6) \qquad [w(x), w(w(x)), w(w(w(x))), w(w(w(w(x)))), \ldots] = \left[w^1(x), w^2(x), w^3(x), w^4(x), \ldots \right]$$

have the form

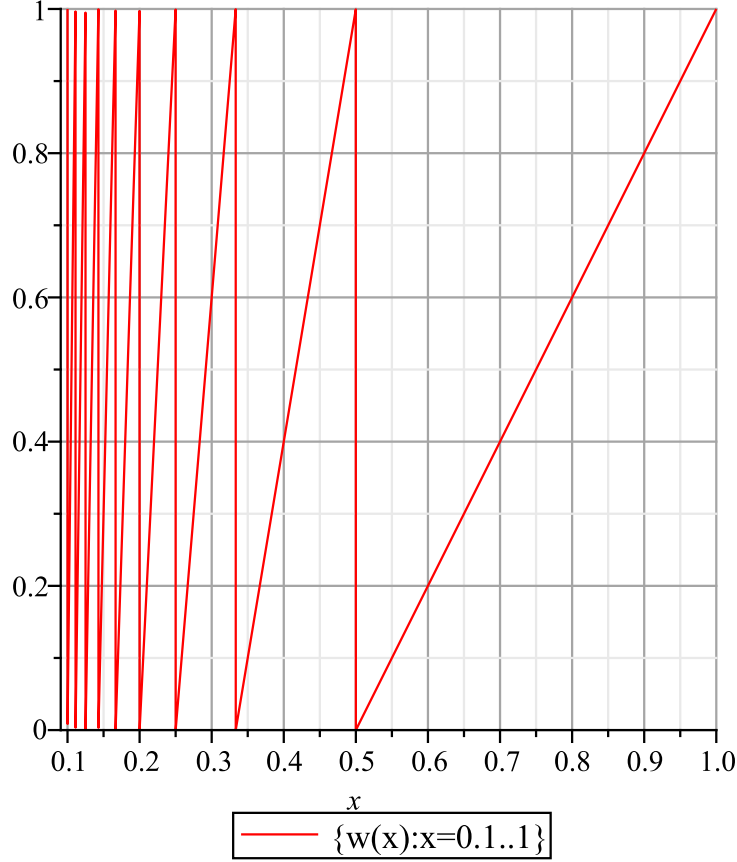$$(7) \qquad\qquad w^r(x) \quad = a_r - (x - c)b_r$$

FIGURE 1. The Harmonic Sawtooth map

where $\{a_r, b_r \in \mathbb{Z} : r \in \mathbb{N}\}$ is a pair of integer sequences and $c \in \mathbb{R}$ is some constant. The sequence of quotients $\frac{a_r}{b_r}$ converges rather quickly to the fixed value

$$(8) \qquad \lim_{r \to \infty} \frac{a_r}{b_r} = x - c \ \ \forall x \in [0, 1]$$

The explicit equation(sometimes called Schröder's equation) for the fixed points of $w(x)$, is

$$(9) \qquad \begin{aligned} \text{Fix}_w^n \ \ &= \{x : w_n(x) = x\} \\ &= \{x : n(xn + x - 1)\chi(x, I_n^H) = x\} \\ &= \{x : n(xn + x - 1) = x\} \\ &= \lim_{x \to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}x^n} \text{Fix}_w(x)}{n!} \\ &= \lim_{x \to 0} \frac{x^{1-n}}{n!} G_{4,4}^{1,4}\left(-x \middle| \begin{matrix} -1 & -1 & -\phi & \phi - 1 \\ 0 & n-1 & 2-\phi & -1-\phi \end{matrix}\right) \\ &= \frac{n}{n^2+n-1} \end{aligned}$$

where G is the Meijer-G function and $\text{Fix}_w(x)$ is the generation function

$$
\begin{aligned}
\text{Fix}_w(x) &= \sum_{n=1}^{\infty} \text{Fix}_w^n\, x^n \\
&= \sum_{n=1}^{\infty} \frac{nx^n}{n^2+n-1} \\
&= \tfrac{1}{10}\left((5-\sqrt{5})\Phi\left(x,1,1-\phi\right) + (5-\sqrt{5})\Phi\left(x,1,\phi\right)\right)
\end{aligned}
\tag{10}
$$

where $\Phi(z,a,v)$ is the Lerch Transcendent (159), and ϕ is the Golden Ratio, which is the ratio of two numbers having the property that the ratio of the sum to the larger equals the ratio of the larger to the smaller. [7, Ch.XX][35, I.7][16, p.50][28] The number ϕ can be called the "most irrational number" because its continued fraction expansion, given by iterations of the Gauss map (78), converges more slowly than any other number. The constant ϕ satisfies the simple identities

$$
\phi = \frac{1}{\phi - 1} = \frac{\phi}{\phi + 1}
\tag{11}
$$

$$
\phi^2 - \phi - 1 = 0
\tag{12}
$$

An interesting fact is that the density of a motif in a certain noncommutative space described in [23, 5.1] must necessarily be an element of the group $\mathbb{Z} + \phi\mathbb{Z}$.

1.2. Integrals Transforms of w(x).

1.2.1. *Dirichlet Polynomial Series and the Mellin Transform of $w(x)$.* The Mellin transform of the harmonic saw map $w(x)$, multiplied by

$$
\tau(s) = s\frac{s+1}{s-1}
\tag{13}
$$

is an analytic continuation of the Riemann zeta function $\zeta(s)\forall - \text{Re}(s) \notin \mathbb{N}$. This form of the zeta function, denoted by $\zeta_w(s)$, is the infinite sum of the Mellin transformations of the component functions.

$$
\begin{aligned}
M[w_n(x); x \to s] &= \int_{\frac{1}{n+1}}^{\frac{1}{n}} w_n(x)x^{s-1}\mathrm{d}x \\
&= \int_{n}^{n+1} w_n(x^{-1})x^{-s-1}\mathrm{d}x \\
&= \int_0^1 n(xn + x - 1)\chi(x, I_n^H)x^{s-1}\mathrm{d}x \\
&= \int_{\frac{1}{n+1}}^{\frac{1}{n}} n(xn + x - 1)x^{s-1}\mathrm{d}x \\
&= -\frac{e^{\ln(n)(1-s)} - ne^{s\ln\left(\frac{1}{n+1}\right)} - se^{s\ln\left(\frac{1}{n}\right)}}{s^2+s} \\
&= \frac{n^s n + s(n+1)^s - (n+1)^s n}{n^s s(n+1)^s + n^s s^2(n+1)^s} \\
&= -\frac{n^{1-s} - n(n+1)^{-s} - sn^{-s}}{s^2+s}
\end{aligned}
\tag{14}
$$

There is a conjugate pair of inverse branches of $\tau(s)$ found by solving

$$
\begin{aligned}
\tau_{\pm}^{-1}(t) &= \{s : \tau(s) = t\} \\
&= \left\{s : s\frac{s+1}{s-1} = t\right\} \\
&= \tfrac{t}{2} - \tfrac{1}{2} \pm \frac{\sqrt{1 - 6t + t^2}}{2}
\end{aligned}
\tag{15}
$$

where $\tau_+^{-1}(t)$ and $\tau_-^{-1}(t)$ denote the positive and negative solutions respectively. The coefficients in the series expansions are integers enumerating the *large Schröder numbers* S_n which count the

number of perfect matchings in a triangular grid of n squares, named after Ernst Schröder(1841-1902).[17, A006318][4, p.340][10]

$$(16) \qquad S_n = {}_2F_1\left(\begin{array}{cc} n+1 & 2-n \\ 2 & \end{array} \middle| -1\right)2$$

where ${}_pF_q$ is a hypergeometric function. We have

$$(17) \qquad \begin{aligned} \lim_{t\to 0} \frac{\frac{\mathrm{d}^n}{\mathrm{d}t^n}\tau_+^{-1}(t)}{n!} &= \begin{cases} 0 & n=0 \\ -1 & n=1 \\ -S_n & n\geqslant 2 \end{cases} \\ \lim_{t\to 0} \frac{\frac{\mathrm{d}^n}{\mathrm{d}t^n}\tau_-^{-1}(t)}{n!} &= \begin{cases} -1 & n=0 \\ 2 & n=1 \\ S_n & n\geqslant 2 \end{cases} \end{aligned}$$

(18)

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
S_n	1	1	2	6	22	90	394	1806	8558	41586	206098	1037718	5293446	27297738	142078746

The residue at the singular point $s=1$ of $\tau(s)$ is

$$(19) \qquad \begin{aligned} \operatorname*{Res}_{s=1}\left(\tau(s)M[w_n(x); x\to s]\right) &= \operatorname*{Res}_{s=1}\left(\frac{s^2+s}{s-1}\left(-\frac{n^{1-s}-n(n+1)^{-s}-sn^{-s}}{s^2+s}\right)\right) \\ &= \operatorname*{Res}_{s=1}\left(\frac{n(n+1)^{-s}-n^{1-s}+sn^{-s}}{s-1}\right) \\ &= 2\int_{\frac{1}{n+1}}^{\frac{1}{n}} w_n(x)\mathrm{d}x \\ &= 2\int_{\frac{1}{n+1}}^{\frac{1}{n}} n(xn+x-1)\mathrm{d}x \\ &= \frac{1}{n^2+n} \end{aligned}$$

The infinite sum of the Mellin transforms multiplied by $\tau(s)$ analytically continues $\zeta(s)\forall(-\operatorname{Re}(s))\notin\mathbb{N}$

$$(20) \qquad \begin{aligned} \zeta_w(s) &= \zeta(s) \\ &= \tau(s)M[w(x); x\to s] \\ &= \tau(s)\int_0^1 w(x)x^{s-1}\mathrm{d}x \\ &= s\frac{s+1}{s-1}\int_0^1 \lfloor x^{-1}\rfloor\left(x\lfloor x^{-1}\rfloor + x - 1\right)x^{s-1}\mathrm{d}x \\ &= s\frac{s+1}{s-1}\sum_{n=1}^\infty M[w_n(x); x\to s] \\ &= s\frac{s+1}{s-1}\sum_{n=1}^\infty \int_{\frac{1}{n+1}}^{\frac{1}{n}} n(xn+x-1)x^{s-1}\mathrm{d}x \\ &= \sum_{n=1}^\infty s\frac{s+1}{s-1}\left(-\frac{n^{1-s}-n(n+1)^{-s}-sn^{-s}}{s(s+1)}\right) \\ &= \sum_{n=1}^\infty \frac{n(n+1)^{-s}-n^{1-s}+sn^{-s}}{s-1} \\ &= \frac{1}{s-1}\sum_{n=1}^\infty n(n+1)^{-s}-n^{1-s}+sn^{-s} \end{aligned}$$

Unlike the Mellin transform of the Gauss map (78), which must be multiplied by the factor s then subtracted from $\frac{s}{s-1}$ before it equals $\zeta(s)$, the "harmonic sawtooth continuation" $\zeta_w(s)$ of the zeta function $\zeta(s)$ has the fortuitous property that it equals $\zeta(s)$ after multiplying $M[w(x); x\to s]$ by $\tau(s)$. This property of $w(x)$ allows us to put $\tau(s)$ inside the sum to get an expression, denoted by $\zeta_w(N; s)$, involving the difference of two Dirichlet polynomials, one of which is scaled by s. The

substition $\infty \to N$ is made in the infinite sum appearing the expression for $\zeta_w(s)$ to get

$$
\begin{aligned}
(21) \quad \zeta_w(N;s) \quad &= \tau(s) \sum_{n=1}^N M[w_n(x); x \to s] \\
&= \frac{1}{s-1} \sum_{n=1}^N n(n+1)^{-s} - n^{1-s} + sn^{-s} \\
&= \frac{1}{s-1} \left(s + (N+1)^{1-s} - 1 + s \sum_{n=2}^N n^{-s} - \sum_{n=2}^{N+1} n^{-s} \right) \\
&= \frac{N}{(s-1)(N+1)^s} - \frac{\cos(\pi s)\Psi(s-1,N+1)}{\Gamma(s)} + \zeta(s) \, \forall s \in \mathbb{N}^*
\end{aligned}
$$

with equality in the limit except at the negative integers

$$
(22) \qquad \lim_{N \to \infty} \zeta_w(N;s) \quad = \zeta(s) \forall - s \notin \mathbb{N}^*
$$

The functions $\zeta_w(N;s)$ have real roots at $s=0$ and $s=-1$. That is

$$
(23) \qquad \lim_{s \to 0} \zeta_w(N;s) = \lim_{s \to -1} \zeta_w(N;s) = 0
$$

The residue of $\zeta_w(N;s)$ at $s=1$ is given by

$$
\begin{aligned}
(24) \quad \operatorname*{Res}_{s=1}(\zeta_w(N;s)) \quad &= \operatorname*{Res}_{s=1}\left(\tau(s) \sum_{n=1}^N M[w_n(x); x \to s] \right) \\
&= \sum_{n=1}^N \operatorname*{Res}_{s=1}\left(\tau(s) M[w_n(x); x \to s] \right) \\
&= \sum_{n=1}^N \frac{1}{n^2+n} \\
&= \frac{N}{N+1}
\end{aligned}
$$

Thus, as required

$$
\begin{aligned}
(25) \quad \lim_{N \to \infty} \operatorname*{Res}_{s=1}(\zeta_w(N;s)) \quad &= \lim_{N \to \infty} \sum_{n=1}^N \frac{1}{n^2+n} \\
&= \lim_{N \to \infty} \frac{N}{N+1} \\
&= 1
\end{aligned}
$$

The function $\tau(s)$ has zeros at -1 and 0 and a simple pole at $s=1$ with residue

$$
(26) \qquad \operatorname*{Res}_{s=1}(\tau(s)) \quad = \operatorname*{Res}_{s=1}\left(s\frac{s+1}{s-1} \right) = 2
$$

The Mellin transform of $\tau(s)$ has an interesting Laurent series, convergent on the unit disc, given by

$$
\begin{aligned}
(27) \quad M[\tau(s); s \to t] \quad &= \int_0^\infty \tau(s) s^{t-1} \mathrm{d}s \\
&= \int_0^\infty s\frac{s+1}{s-1} s^{t-1} \mathrm{d}s \\
&= \sum_{n=1}^\infty 4\zeta(2n-2)t^{2n-3} \quad \forall |t| < 1 \\
&= -4i\pi \frac{\left(\frac{1}{2}e^{i\pi t} + \frac{1}{2}e^{-i\pi t} \right)}{e^{i\pi t} - e^{-i\pi t}} \\
&= -2\pi \frac{\cos(\pi t)}{\sin(\pi t)}
\end{aligned}
$$

The transformations $M[w_n(x); x \to s]$ have removable singularities at -1 and 0 where the limits are given by

$$
\begin{aligned}
(28) \quad \lim_{s \to -1} M[w_n(x); x \to s] \quad &= n^2 \ln\left(\frac{n+1}{n} \right) + n \ln\left(\frac{n+1}{n} \right) - n \\
\lim_{s \to 0} M[w_n(x); x \to s] \quad &= 1 + n \ln\left(\frac{n+1}{n} \right)
\end{aligned}
$$

(29)
$$\lim_{s\to 0} e^{-M[w_n(x);x\to s]} = (n+1)^n n^{-n} e^{-1}$$
$$\lim_{s\to -1} e^{-M[w_n(x);x\to s]} = n^{(n^2+n)}(n+1)^{(-n^2-n)}e^n$$

So, (14) can be rewritten as

(30)
$$M[w_n(x);x\to s] = \begin{cases} n^2 \ln\left(\frac{n+1}{n}\right) + n\ln\left(\frac{n+1}{n}\right) - n & s=-1 \\ 1 + n\ln\left(\frac{n+1}{n}\right) & s=0 \\ -\frac{n^{1-s}-n(n+1)^{-s}-sn^{-s}}{s^2+s} & \text{otherwise} \end{cases}$$

Furthermore, we have the limits

(31)
$$\lim_{n\to\infty}\lim_{s\to -1} M[w_n(x);x\to s] = \lim_{n\to\infty}\lim_{s\to -1} -\frac{n^{1-s}-n(n+1)^{-s}-sn^{-s}}{s^2+s}$$
$$= \lim_{n\to\infty} n^2 \ln\left(\frac{n+1}{n}\right) + n\ln\left(\frac{n+1}{n}\right) - n$$
$$= \frac{1}{2}$$

and

(32)
$$\lim_{n\to\infty}\lim_{s\to 0} M[w_n(x);x\to s] = \lim_{n\to\infty}\lim_{s\to 0} -\frac{n^{1-s}-n(n+1)^{-s}-sn^{-s}}{s^2+s}$$
$$= \lim_{n\to\infty} 1 + n\ln\left(\frac{n+1}{n}\right)$$
$$= 0$$

The sum of limits over n at $s=-1$ is Euler's constant. [13][14, 1.1]

(33)
$$\sum_{n=1}^{\infty} \frac{M[w_n(x);x\to -1]}{n} = \sum_{n=1}^{\infty} \frac{1+n\ln\left(\frac{n+1}{n}\right)}{n}$$
$$= \lim_{s\to 1} \zeta(s) - \frac{1}{s-1}$$
$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \ln(n)$$
$$= \gamma$$
$$\cong 0.577215664901533\ldots$$

1.2.2. *The Reflection Formula for $\zeta_w(N;s)$.* There is a reflection equation for the finite-sum approximation $\zeta_w(N;s)$ which is similiar to the well-known formula $\zeta(1-s) = \chi(s)\zeta(s)$ with $\chi(s) = 2(2\pi)^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)$. The solution to

(34)
$$\zeta_w(N;1-s) = \chi(N;s)\zeta_w(N;s)$$

is given by the expression

(35)
$$\chi(N;s) = \frac{\zeta_w(N;1-s)}{\zeta_w(N;s)}$$
$$= \frac{\sum_{n=1}^{N} -\frac{-n^s+(n+1)^{s-1}n+n^{s-1}-n^{s-1}s}{s}}{\sum_{n=1}^{N} \frac{-n^{1-s}+(n+1)^{-s}n+n^{-s}s}{s-1}}$$
$$= -\frac{(s-1)\sum_{n=1}^{N} -n^s+(n+1)^{s-1}n+n^{s-1}-n^{s-1}s}{s\sum_{n=1}^{N} -n^{1-s}+(n+1)^{-s}n+n^{-s}s}$$

which satisfies

(36)
$$\chi(N;1-s) = \chi(N;s)^{-1}$$

The functions $\chi(N;s)$, indexed by N, have singularities at $s=0$. Let

(37)
$$a(N) = \sum_{n=1}^{N} n(\ln(n+1)-\ln(n))$$
$$b(N) = \sum_{n=1}^{N} \frac{\ln(n)n^2-\ln(n+1)n^2-\ln(n)}{n(n+1)}$$
$$c(N) = \frac{1}{2}\sum_{n=1}^{N} n\left(\ln(n+1)^2-\ln(n)^2\right)$$

then the residue at the singular point $s = 0$ is given by the expression

(38)
$$\operatorname*{Res}_{s=0}(\chi(N;s)) = -\operatorname*{Res}_{s=1}(\chi(N;s)^{-1})$$
$$= \frac{1+\gamma+\Psi(n+2)-\frac{2}{N+1}+b(N)-\frac{N(\ln(\Gamma(N+1))-c(N))}{(N-a(N))(N+1)}}{a(N)-N}$$
$$= \frac{1+\gamma+\Psi(n+2)-\frac{2}{N+1}+\sum_{n=1}^{N}\frac{\ln(n)n^2-\ln(n+1)n^2-\ln(n)}{n(n+1)}-\frac{N\left(\ln(\Gamma(N+1))-\frac{1}{2}\sum_{n=1}^{N}n\left(\ln(n+1)^2-\ln(n)^2\right)\right)}{\left(N-\sum_{n=1}^{N}n(\ln(n+1)-\ln(n))\right)(N+1)}}{\left(\sum_{n=1}^{N}n(\ln(n+1)-\ln(n))\right)-N}$$

which has the limit

(39)
$$\lim_{N\to\infty}\operatorname*{Res}_{s=0}(\chi(N;s)) = 1$$

We also have the residue of the reciprocal at $s = 2$

(40)
$$\operatorname*{Res}_{s=2}(\chi(N;s)^{-1}) = \frac{\frac{2N}{(N+1)^2}-2\Psi(1,N+1)+2\zeta(2)}{\frac{(N+1)^2}{2}-\frac{N}{2}-\frac{1}{2}-\sum_{n=1}^{N}n(\ln(n+1)+\ln(n+1)n-\ln(n)-n\ln(n))}$$

which vanishes as N tends to infinity

(41)
$$\lim_{N\to\infty}\operatorname*{Res}_{s=2}(\chi(N;s)^{-1}) = 0$$

As can be seen in the figures below, the residue at $s = 0$ changes sign from negative to positive between the values of $N = 176$ and $N = 177$.
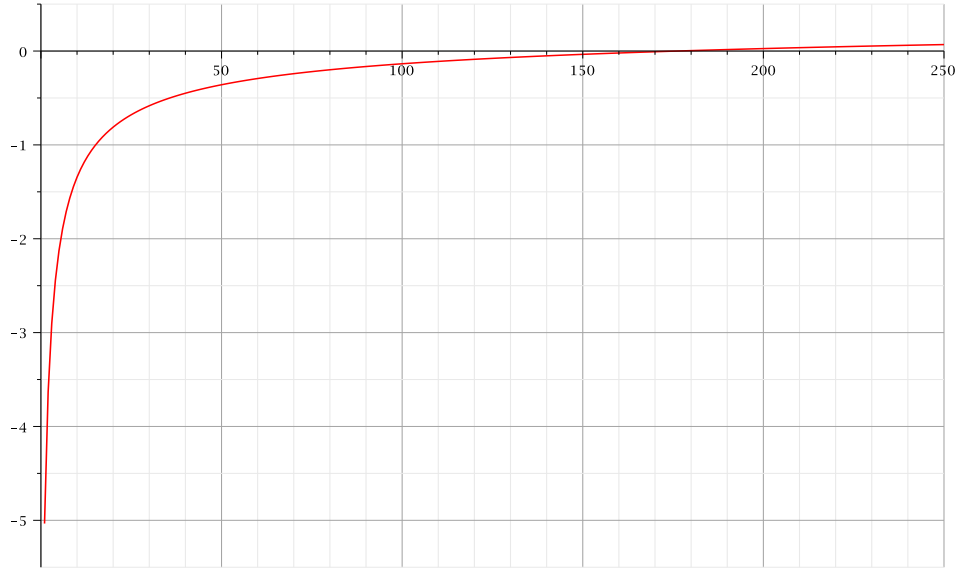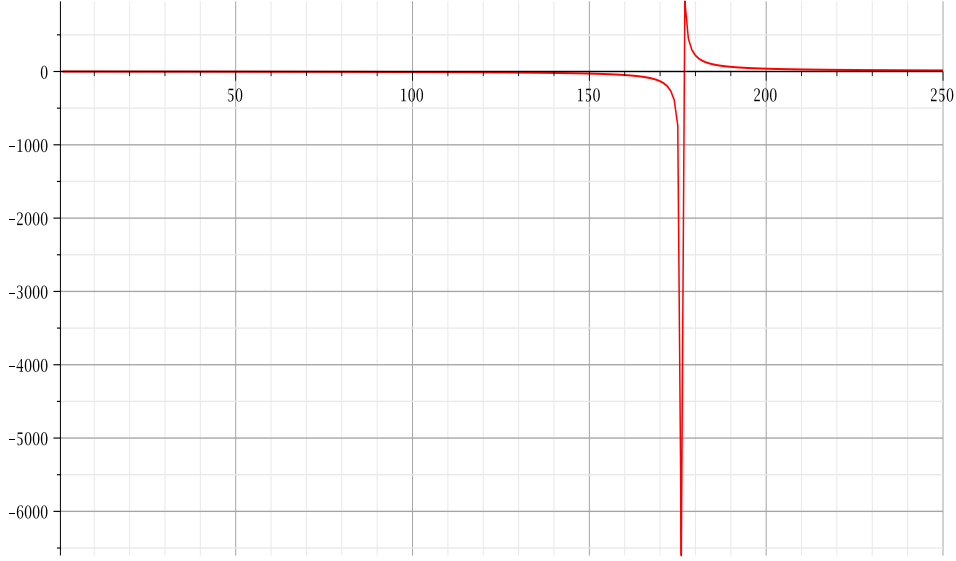

FIGURE 2. $\left\{\operatorname*{Res}_{s=0}(\chi(N;s)) : N = 1\ldots 250\right\}$

FIGURE 3. $\left\{ \operatorname{Res}_{s=0}(\chi(N;s))^{-1} : N = 1\ldots 250 \right\}$

For any positive integer N, we have the limits

(42)
$$
\begin{aligned}
\lim_{s\to 0} \chi(N;s) &= \infty \\
\lim_{s\to 0} \tfrac{\mathrm{d}^n}{\mathrm{d}s^n}\chi(N;s) &= \infty \\
\lim_{s\to \frac{1}{2}} \chi(N;s) &= 1 \\
\lim_{s\to 1} \chi(N;s) &= 0 \\
\lim_{s\to 2} \chi(N;s) &= 0 \\
\lim_{s\to 1} \tfrac{\mathrm{d}}{\mathrm{d}s}\chi(N;s) &= 0
\end{aligned}
$$

The line $\operatorname{Re}(s) = \frac{1}{2}$ has a constant modulus

(43)
$$
\left| \chi\left(N; \frac{1}{2} + is\right) \right| = 1
$$

There is also the complex conjugate symmetry

(44)
$$
\chi(N; x + iy) = \overline{\chi(N; x - iy)}
$$

If $s = n \in \mathbb{N}^*$ is a positive integer then $\chi(N;n)$ can be written as

(45)
$$
\begin{aligned}
\chi(N;n) &= \frac{\zeta_w(N;1-n)}{\zeta_w(N;n)} \\
&= \frac{\sum_{m=1}^{N} - \sum_{k=1}^{n-2} \frac{m^k}{n}\binom{n-1}{k-1}}{\frac{N}{(n-1)(N+1)^n} - \frac{\cos(\pi n)\Psi(n-1,N+1)}{\Gamma(n)} + \zeta(n)} \\
&= \frac{-\sum_{m=1}^{N} \frac{1}{n}\left((n-1)m^{n-1} + m^n - (m+1)^{n-1}m\right)}{\frac{N}{(n-1)(N+1)^n} - \frac{\cos(\pi n)\Psi(n-1,N+1)}{\Gamma(n)} + \zeta(n)}
\end{aligned}
$$

The Bernoulli numbers[1] make an appearance since

(46)
$$
\chi(N;2n)\,\zeta_w(N;2n) = B_{2n}(N+1)^2 \frac{(2n+1)}{2} + \ldots
$$

The denominator of $\chi(N;n)$ has the limits

$$(47) \qquad \begin{aligned} \lim_{N\to\infty} \zeta_w(N;n) &= \zeta(n) \\ \lim_{n\to\infty} \zeta_w(N;n) &= 1 \end{aligned}$$

Another interesting formula gives the limit at $s=1$ of the quotient of successive functions

$$(48) \qquad \begin{aligned} \lim_{s=1} \frac{\chi(N+1;s)}{\chi(N;s)} &= \frac{(N+2)N(N+1-a(N+1))}{(N+1)^2(N-a(N))} \\ &= \frac{(N+2)N\left(N+1-\sum_{n=1}^{N+1} n(\ln(n+1)-\ln(n))\right)}{(N+1)^2\left(N-\sum_{n=1}^{N} n(\ln(n+1)-\ln(n))\right)} \end{aligned}$$
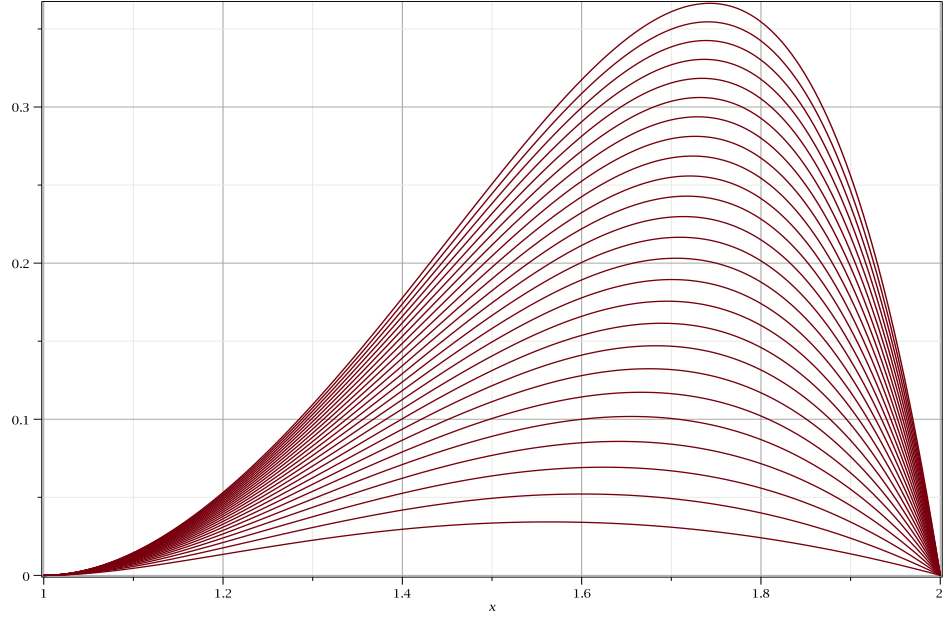

FIGURE 4. $\{\chi(N;s) : s = 1\ldots 2, N = 1\ldots 25\}$

Let

$$(49) \qquad \nu(s) = \chi(\infty;s) = \frac{\zeta(1-s)}{\zeta(s)}$$

Then the residue at the even negative integers is

$$(50) \qquad \operatorname*{Res}_{s=-n}(\nu(s)) = \begin{cases} \frac{\zeta(1-n)}{\frac{d}{ds}\zeta(s)|_{s=-n}} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

1.2.3. *The Laplace Transforms $L[w_n(x); x \to s]$.* The Laplace transform $L[w_n(x); x \to s]$ and its roots are calculated to shed light on the behaviour roots of the Mellin transforms $M[w_n(x); x \to s]$ but it is unclear whether this is accomplished. The Laplace transform (135) of the n-th component
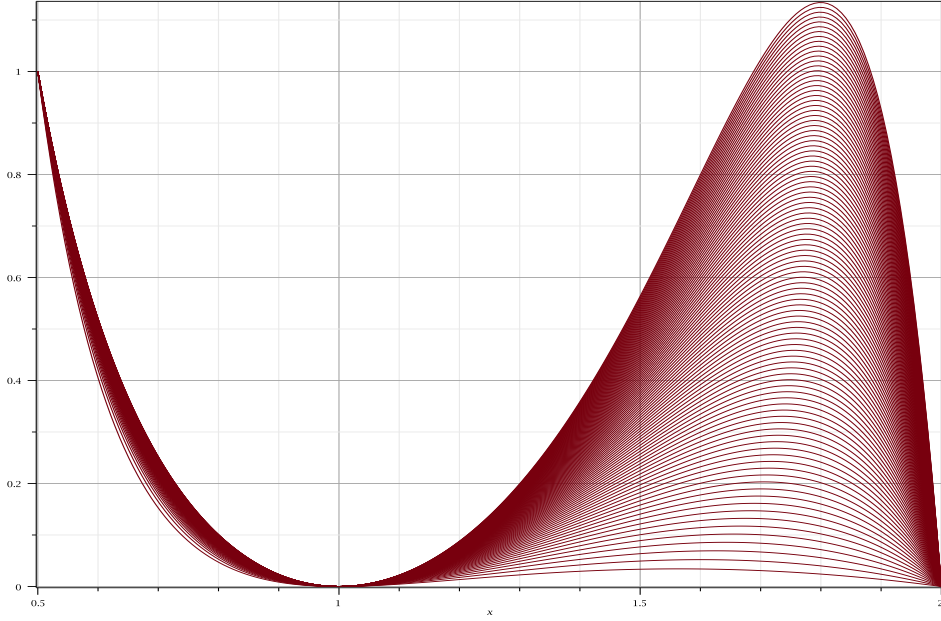
FIGURE 5. $\left\{\chi\left(N;s\right): s = \frac{1}{2}\ldots 2, N = 1\ldots 100\right\}$

function is given by

$$
(51) \quad
\begin{aligned}
L[w_n(x); x \to s] &= \int_0^1 w_n(x) e^{-xs} \mathrm{d}x \\
&= \int_0^1 n(xn + x - 1)\chi(x, I_n^H) e^{-xs} \mathrm{d}x \\
&= \int_{\frac{1}{n+1}}^{\frac{1}{n}} n(xn + x - 1) e^{-xs} \mathrm{d}x \\
&= \frac{n(n+1)e^{-\frac{s}{n+1}} - (n^2 + n + s)e^{-\frac{s}{n}}}{s^2}
\end{aligned}
$$

There is a removable singularity at $s = 0$ which has the limit

$$
(52) \quad
\begin{aligned}
L[w_n(x); x \to 0] &= \lim_{s \to 0} L[w_n(x); x \to s] \\
&= \lim_{s \to 0} \frac{n(n+1)e^{-\frac{s}{n+1}} - (n^2 + n + s)e^{-\frac{s}{n}}}{s^2} \\
&= \frac{1}{2}\operatorname*{Res}_{s=1}\left(\tau(s) M[w_n(x); x \to s]\right) \\
&= \frac{1}{2n(n+1)}
\end{aligned}
$$

Additionally,

$$
(53) \quad
\begin{aligned}
\sum_{n=1}^{\infty} L[w_n(x); x \to 0] &= \sum_{n=1}^{\infty} \lim_{s \to 0} \frac{n(n+1)e^{-\frac{s}{n+1}} - (n^2 + n + s)e^{-\frac{s}{n}}}{s^2} \\
&= \sum_{n=1}^{\infty} \frac{1}{2n(n+1)} \\
&= \lim_{n \to \infty} \lim_{s \to -1} M[w_n(x); x \to s] \\
&= \lim_{n \to \infty} \lim_{s \to -1} -\frac{n^{1-s} - n(n+1)^{-s} - sn^{-s}}{s^2 + s} \\
&= \frac{1}{2}
\end{aligned}
$$

The roots of $L[w_n(x); x \to s]$ are enumerated by

$$(54) \qquad \rho_{w_n}^L(m) \quad = \{s : L[w_n(x); x \to s] = 0\} = -n(n+1)W(m, -e^{-1}) - n(n+1)$$

where $W(m, x)$ is the Lambert W function (146) and $m \in \mathbb{Z}, n \in \mathbb{N}$. It can be verified that

$$(55) \qquad L[w_n(z); x \to \rho_{w_n}^L(m)] \quad = \frac{e^{(n+1)(1+W(m,-e^{-1}))}W(m,-e^{-1})+e^{n(1+W(m,-e^{-1}))}}{n(1+W(m,-e^{-1}))^2(n+1)} = 0$$

where $-e^{-1}$ is expressed as the continued fraction via its quotient sequence

$$(56) \quad -e^{-1} = [-1, 1, 1, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, 14, 1, 1, 16, 1, 1, 18, 1, 1, 20, ...]$$

The roots $\rho_{w_n}^L(m)$ satisfy a functional reflection equation with respect to m

$$(57) \qquad \begin{array}{ll} \rho_{w_n}^L(m) & = \overline{\rho_{w_n}^L(-m-1)} \\ -n(n+1)W(m, -e^{-1}) - n(n+1) & = -n(n+1)\overline{W(-m-1, -e^{-1})} - n(n+1) \end{array}$$

where $\bar{x} = \mathrm{Re}(x) - \mathrm{Im}(x)$ denotes complex conjugation. The quotients of the roots of consecutive transforms is

$$(58) \qquad \frac{\rho_{w_n}^L(m)}{\rho_{w_{n-1}}^L(m)} \quad = \frac{n+1}{n-1}$$

Thus

$$(59) \qquad \lim_{n \to \infty} \frac{\rho_{w_n}^L(m)}{\rho_{w_{n-1}}^L(m)} \quad = \lim_{n \to \infty} \frac{n+1}{n-1} = 1$$

1.2.4. *The Roots* $\rho_{w_n}^M(m)$ *of* $M[w_n(x); x \to s]$. Define

$$(60) \qquad \begin{array}{ll} M_{\tau w}(s, n) & = \tau(s)M[w_n(x); x \to s] \\ & = \frac{n(n+1)^{-s}-n^{1-s}+sn^{-s}}{s-1} \end{array}$$

as in (21) and its infinte number of inverse branches (which are currently lacking closed-form expression if such a thing is possible except when $n = 1$), where the branches are indexed by m

$$(61) \qquad \begin{array}{ll} M_{\tau w}^{-1}(z, n, m) & = \{s : \tau(s)M[w_n(x); x \to s] = z\} \\ & = \left\{s : \frac{n(n+1)^{-s}-n^{1-s}+sn^{-s}}{s-1} = z\right\} \end{array}$$

then we see that the first function where $n = 1$, $M_{\tau w}^{-1}(z, 1, m)$, has the closed-form

$$(62) \qquad M_{\tau w}^{-1}(z, 1, m) \quad = \frac{W\left(m, \frac{\ln(\sqrt{2})}{z-1}\right)+\ln(2)}{\ln(2)}$$

where W is the Lambert W function (146). It is verified that

$$(63) \qquad M_{\tau w}(M_{\tau w}^{-1}(z, 1, m), 1, m) \quad = -\frac{1-2^{-\frac{W\left(m, \frac{\ln(\sqrt{2})}{z-1}\right)+\ln(2)}{\ln(2)}}-\frac{W\left(m, \frac{\ln(\sqrt{2})}{z-1}\right)+\ln(2)}{\ln(2)}}{\frac{W\left(m, \frac{\ln(\sqrt{2})}{z-1}\right)+\ln(2)}{\ln(2)}-1} = z$$

Furthermore, let $\rho_{w_n}^M(m)$ denote the m-th root of $M[w_n(x); x \to s]$

$$(64) \qquad \begin{array}{ll} \rho_{w_n}^M(m) & = \{s : M[w_n(x); x \to s] = 0\} \\ & = M_w^{-1}(0, n, m) \end{array}$$

which satisfies

$$(65) \qquad \mathrm{Im}\left(\rho_{w_n}^M(m)\right) > \mathrm{Im}\left(\rho_{w_{n-1}}^M(m)\right)$$
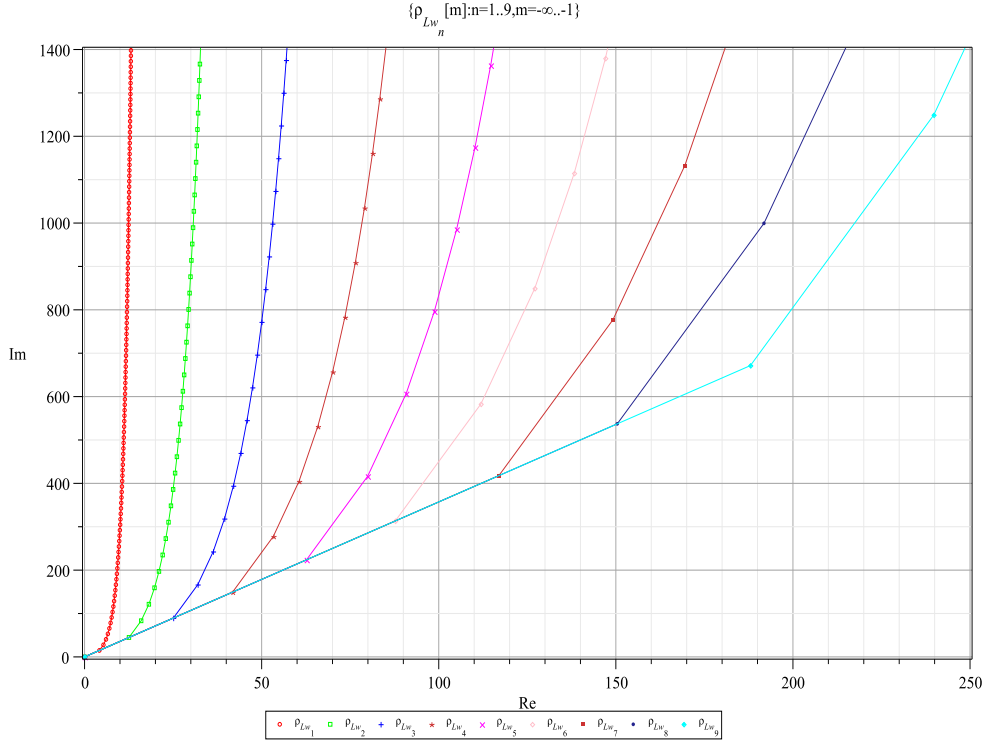
FIGURE 6. The Roots $\rho_{w_n}^L(m)$ of $\{L[w_n(x); x \to s] : n \in 1 \dots 9\}$

$$\lim_{m \to \pm\infty} \operatorname{Re}(\rho_{w_n}^M(m)) \quad = 0 \tag{66}$$

Thus,

$$\lim_{m \to \pm\infty} \arg\left(\rho_{w_n}^M(m)\right) \quad = \frac{\pi}{2} \tag{67}$$

1.2.5. *Quotients and Differences of $\rho_{w_n}^M(m)$.* Let

$$\Delta\rho_{w_n}^M(m) \quad = \rho_{w_n}^M(m+1) - \rho_{w_n}^M(m) \tag{68}$$

be the forward difference of consecutive roots of $M[w_n(x); x \to s]$. The limiting difference between consecutive roots is the countably infinite set of solutions to the equation $n^{\frac{s}{2}} + (n+1)^{\frac{s}{2}} = 0$ given

by

$$
\begin{aligned}
\Delta\rho^M_{w_n}(\pm\infty) &= \lim_{m\to\pm\infty}\Delta\rho^M_{w_n}(m) \\
&= \left\{s : n^{\frac{s}{2}} + (n+1)^{\frac{s}{2}} = 0\right\} \\
&= \lim_{m\to\pm\infty}\left(\rho^M_{w_n}(m+1) - \rho^M_{w_n}(m)\right) \\
&= \lim_{m\to\pm\infty}\left(\frac{\rho^M_{w_n}(m)}{m}\right) \\
&= \frac{2\pi i}{M[\chi(x,I^H_n);x\to 0]} \\
&= \frac{2\pi i}{\lim_{s\to 0}\left(\frac{n^{-s}-(n+1)^{-s}}{s}\right)} \\
&= \frac{2\pi i}{\ln(n+1)-\ln(n)}
\end{aligned}
$$

(69)

Let $\mathcal{Q}^\infty_{\rho^M_{w_n}}$ denote the limit

$$
\begin{aligned}
\mathcal{Q}^\infty_{\rho^M_{w_n}} &= \lim_{m\to\pm\infty}\frac{\rho^M_{w_n}(m)}{\rho^M_{w_{n-1}}(m)} \\
&= \frac{\Delta\rho^M_{w_n}(\pm\infty)}{\Delta\rho^M_{w_{n-1}}(\pm\infty)} \\
&= \frac{M\left[\chi\left(x,\left(\frac{1}{n+1},\frac{1}{n}\right)\right);x\to 0\right]}{M\left[\chi\left(x,\left(\frac{1}{n},\frac{1}{n-1}\right)\right);x\to 0\right]} \\
&= \frac{\ln(n)-\ln(n-1)}{\ln(n+1)-\ln(n)}
\end{aligned}
$$

(70)

then we also have the limit of the limits $\mathcal{Q}^\infty_{\rho^M_{w_n}}$ as $n\to\infty$ given by

$$
\begin{aligned}
\lim_{n\to\pm\infty}\mathcal{Q}^\infty_{\rho^M_{w_n}} &= \lim_{n\to\pm\infty}\left(\frac{\Delta\rho^M_{w_n}(\pm\infty)}{\Delta\rho^M_{w_{n-1}}(\pm\infty)}\right) \\
&= \lim_{n\to\pm\infty}\left(\frac{\frac{\ln(n)-\ln(n-1)}{\ln(n+1)-\ln(n)}}{\frac{\ln(n-1)-\ln(n-2)}{\ln(n)-\ln(n-2)}}\right) \\
&= \lim_{n\to\pm\infty}\left(\frac{\ln(n)-\ln(n-1)}{\ln(n+1)-\ln(n)}\right) \\
&= 1
\end{aligned}
$$

(71)

The limiting quotients $\frac{e^{-\rho^M_{w_n}(m+1)}}{e^{-\rho^M_{w_n}(m)}} = e^{\rho^M_{w_n}(m)-\rho^M_{w_n}(m+1)}$ as $m\to\infty$ are given by

(72)

$$
\begin{aligned}
\lim_{m\to\infty}e^{\rho^M_{w_n}(m)-\rho^M_{w_n}(m+1)} &= 1 - 2\sin\left(\frac{\pi}{\ln(n+1)-\ln(n)}\right)^2 - 2i\cos\left(\frac{\pi}{\ln(n+1)-\ln(n)}\right)\sin\left(\frac{\pi}{\ln(n+1)-\ln(n)}\right) \\
&= e^{-\frac{2\pi i}{\ln(n+1)-\ln(n)}}
\end{aligned}
$$

where we have

$$
\begin{aligned}
\left|\lim_{m\to\infty}e^{\rho^M_{w_n}(m)-\rho^M_{w_n}(m+1)}\right| &= \lim_{m\to\infty}\left|e^{\rho^M_{w_n}(m)-\rho^M_{w_n}(m+1)}\right| \\
&= \sqrt{e^{-\frac{2\pi i}{\ln(n+1)-\ln(n)}}} \\
&= 1
\end{aligned}
$$

(73)

and

(74)
$$
\lim_{n\to\infty}\lim_{m\to\infty}e^{\rho^M_{w_n}(m)-\rho^M_{w_n}(m+1)} = -1
$$
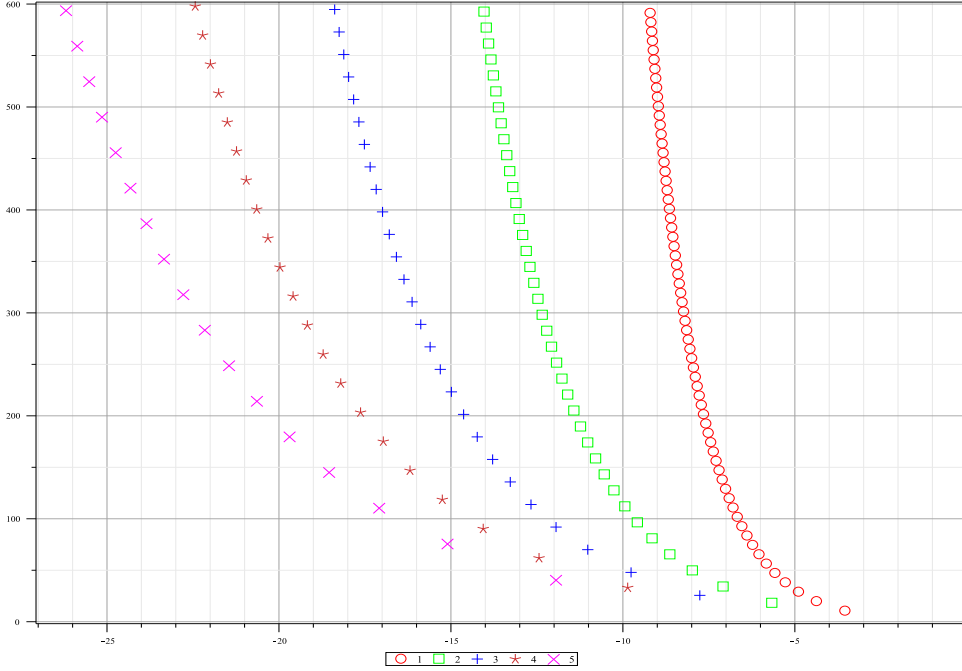
FIGURE 7. $\{\rho_{w_n}^M(m) : m = 1 \dots 5\}$

1.2.6. *The Laplace Transform* $L\left[(s-1)M\left[w_n(x); x \to s\right]; s \to t\right]$. The Laplace transform of $(s-1)M[w_n(x); x \to s]$ defined by

$$
\begin{aligned}
L[(s-1)M[w_n(x); x \to s]; s \to t] \quad &= L[n(n+1)^{-s} + sn^{-s} - n^{1-s}; s \to t] \\
&= \int_0^\infty n(n+1)^{-s} + sn^{-s} - n^{1-s}e^{-st}\mathrm{d}s \\
&= \frac{t+\ln\left(\frac{n^n}{(n+1)^n}\right)t + \ln(n+1) + n\ln(n)^2 - n\ln(n)\ln(n+1)}{(\ln(n)+t)^2(\ln(n+1)+t)}
\end{aligned}
$$

(75)

has poles at $-\ln(n)$ and $-\ln(n+1)$ with residues

(76)
$$
\begin{aligned}
\operatorname*{Res}_{t=-\ln(n)}\Big(L[(s-1)M[w_n(x); x \to s]; s \to t]\Big) \quad &= -n \\
\operatorname*{Res}_{t=-\ln(n+1)}\Big(L[(s-1)M[w_n(x); x \to s]; s \to t]\Big) &= n
\end{aligned}
$$

1.3. The Gauss Map $h(x)$.

1.3.1. Continued Fractions. The Gauss map $h(x)$, also known as the Gauss function or Gauss transformation, maps unit intervals onto unit intervals and by iteration gives the continued fraction expansion of a real number [38, A.1.7][35, I.1][12, X] The n-th component function $h_n(x)$ of the map $h(x)$ is given by

(77)
$$
h_n(x) \quad = \frac{1-xn}{x}\chi\left(x, I_n^H\right)
$$

The infinite sum of the component functions is the Gauss map

$$
\begin{aligned}
h(x) \quad &= \sum_{n=1}^{\infty} h_n(x) \\
&= \sum_{n=1}^{\infty} \frac{1-xn}{x} \chi\left(x, I_n^H\right) \\
&= x^{-1} - \left\lfloor x^{-1} \right\rfloor \\
&= \{x^{-1}\}
\end{aligned}
\tag{78}
$$

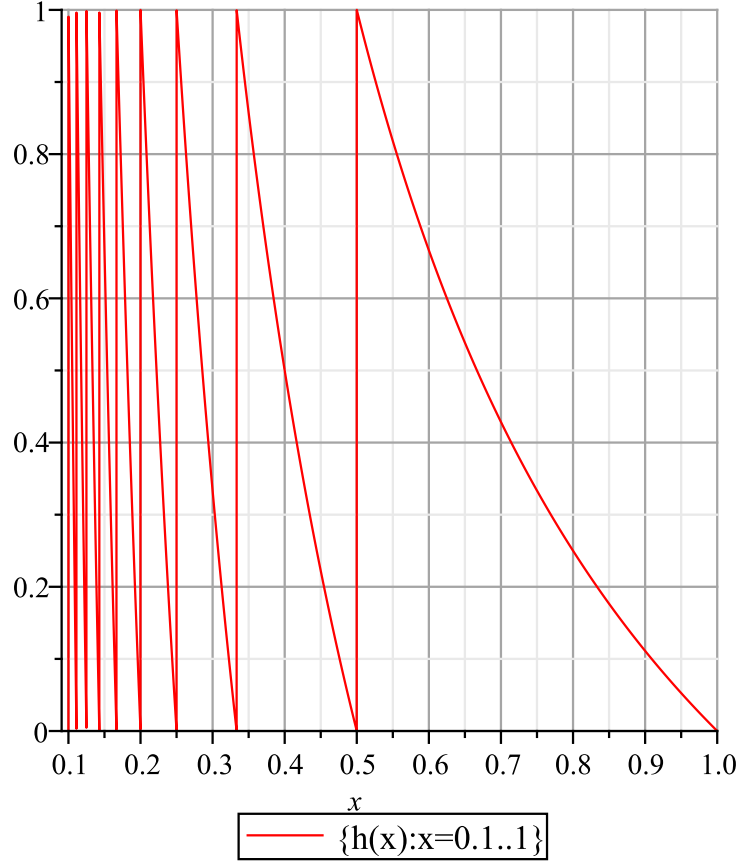

FIGURE 8. The Gauss Map

The fixed points of $h(x)$ are the (positive) solutions to the equation $h_n(x) = x$ given by

$$
\begin{aligned}
\mathrm{Fix}_h^n \quad &= \{x : h_n(x) = x\} \\
&= \left\{x : \frac{1-xn}{x}\chi\left(x, I_n^H\right) = x\right\} \\
&= \left\{x : \frac{1-xn}{x} = x\right\} \\
&= \frac{\sqrt{n^2+4}}{2} - \frac{n}{2}
\end{aligned}
\tag{79}
$$

1.3.2. *The Mellin Transform of $h(x)$.* The Mellin transform (138) of the Gauss map $h(x)$ over the unit interval, scaled by s then subtracted from $\frac{s}{s-1}$, is an analytic continuation of $\zeta(s)$, denoted by $\zeta_h(s)$, valid for all$-\operatorname{Re}(s)) \notin \mathbb{N}$. The transfer operator and thermodynamic aspects of the Gauss map are discussed in [42][41][40][39][36]. The Mellin transform of the n-th component function $w_n(x)$ is given by

$$
\begin{aligned}
(80) \qquad M[h_n(x); x \to s] \quad &= \int_0^1 h_n(x) x^{s-1} \mathrm{d}x \\
&= \int_0^1 \frac{1-xn}{x} \chi\left(x, I_n^H\right) x^{s-1} \mathrm{d}x \\
&= \int_{\frac{1}{n+1}}^{\frac{1}{n}} \left(x^{-1} - \lfloor x^{-1} \rfloor\right) x^{s-1} \mathrm{d}x \\
&= \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{1-xn}{x} x^{s-1} \mathrm{d}x \\
&= -\frac{n(n+1)^{-s} + s(n+1)^{-s} - n^{1-s}}{s^2 - s}
\end{aligned}
$$

which provides an analytic continuation $\zeta_h(s) = \zeta(s) \forall (-\operatorname{Re}(s)) \notin \mathbb{N}$

$$
\begin{aligned}
(81) \qquad \zeta_h(s) \quad &= \frac{s}{s-1} - sM[h(x); x \to s] \\
&= \frac{s}{s-1} - s \int_0^1 h(x) x^{s-1} \mathrm{d}x \\
&= \frac{s}{s-1} - s \int_0^1 \left(x^{-1} - \lfloor x^{-1} \rfloor\right) x^{s-1} \mathrm{d}x \\
&= \frac{s}{s-1} - s \sum_{n=1}^\infty M[h_n(x); x \to s] \\
&= \frac{s}{s-1} - \frac{1}{s-1} \sum_{n=1}^\infty -(n(n+1)^{-s} + s(n+1)^{-s} - n^{1-s}) \\
&= \frac{s}{s-1} - \frac{1}{s-1} \sum_{n=1}^\infty n^{1-s} - n(n+1)^{-s} - s(n+1)^{-s}
\end{aligned}
$$

1.4. The Harmonic Sawtooth Map w(x) as an Ordinary Fractal String.

1.4.1. *Definition and Length.* Let

$$
(82) \qquad I_n^H \quad = \left(\frac{1}{n+1}, \frac{1}{n}\right)
$$

be the n-th harmonic interval, then $\{w(x) \in \mathcal{L}_w : x \in \Omega\}$ is the piecewise monotone mapping of the unit interval onto itself. The fractal string $\mathcal{L}_w$ associated with $w(x)$ is the set of connected component functions $w_n(x) \subset w(x)$ where each $w_n(x)$ mapsI_n^H onto $(0,1)$ and vanishes when $x \notin I_n^H$. Thus, the disjoint union of the connected components of $\mathcal{L}_w$ is the infinite sum $w(x) = \sum_{n=1}^\infty w_n(x)$ where only 1 of the $w_n(x)$ is nonzero for each x, thus $w(x)$ maps entire unit interval onto itself uniqely except except for the points of discontinuity on the boundary $\partial \mathcal{L}_w = \{0, \frac{1}{n} : n \in \mathbb{N}^*\}$ where a choice is to be made between 0 and 1 depending on the direction in which the limit is approached. Let

$$
(83) \qquad w_n(x) \quad = n(xn + x - 1)\chi(x, I_n^H)
$$

where $\chi(x, I_n^H)$ is the n-th harmonic interval indicator (127)

$$
(84) \qquad \chi(x, I_n^H) \quad = \theta\left(\frac{xn+x-1}{n+1}\right) - \theta\left(\frac{xn-1}{n}\right)
$$

The substitution $n \to \lfloor \frac{1}{x} \rfloor$ can be made in (132) where it is seen that

$$
(85) \qquad \chi\left(x, I_{\lfloor x^{-1} \rfloor}^H\right) \quad = \theta\left(\frac{x\lfloor x^{-1} \rfloor + x - 1}{\lfloor x^{-1} \rfloor + 1}\right) - \theta\left(\frac{x\lfloor x^{-1} \rfloor - 1}{\lfloor x^{-1} \rfloor}\right) = 1
$$

and so making the same substitution in (83) gives

$$
\begin{aligned}
w(x) \quad &= \textstyle\sum_{n=1}^{\infty} w_n(x) \\
&= \textstyle\sum_{n=1}^{\infty} n(xn + x - 1)\chi(x, I_n^H) \\
&= \left\lfloor x^{-1} \right\rfloor \left(x \left\lfloor x^{-1} \right\rfloor + x - 1 \right) \\
&= x \left\lfloor x^{-1} \right\rfloor^2 + x \left\lfloor x^{-1} \right\rfloor - \left\lfloor x^{-1} \right\rfloor
\end{aligned}
$$

(86)

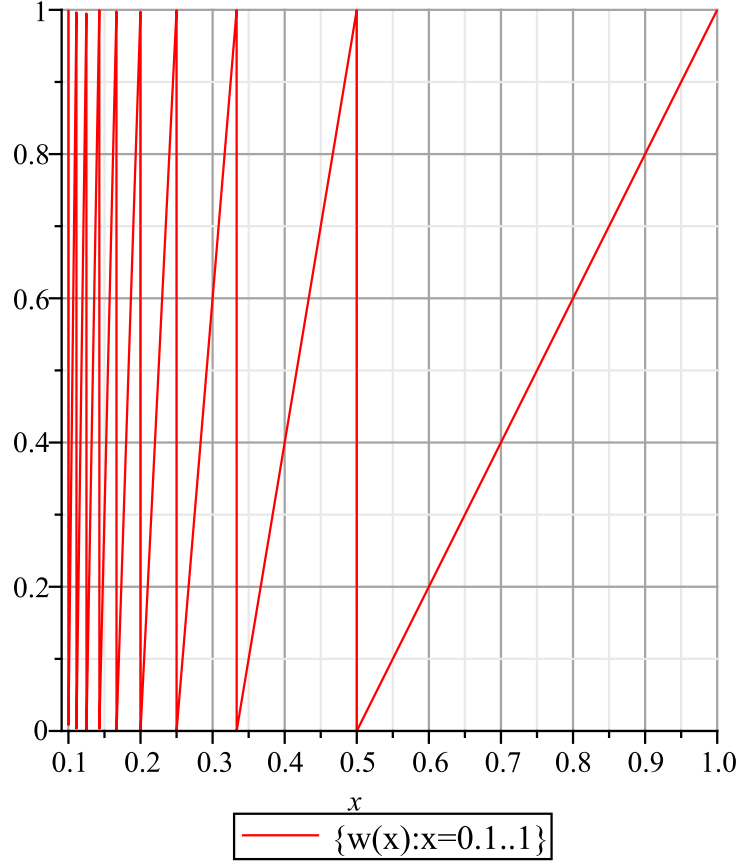FIGURE 9. The Harmonic Sawtooth Map

The intervals I_n^w will be defined such that $\ell w_n = |I_n^w| = |w_n(x)|$. Let

$$
\begin{aligned}
\mathfrak{h}_n \quad &= \int_{I_n^H} x(n+1)n\,\mathrm{d}x \\
&= \frac{\left(\frac{1}{n+1} + \frac{1}{n} \right)}{2} \\
&= \frac{2n+1}{2n(n+1)}
\end{aligned}
$$

(87)

be the midpoint of I_n^H then

$$
\begin{aligned}
(88) \qquad I_n^w \quad &= \left(\mathfrak{h}_n - \tfrac{|w_n(x)|}{2}, \mathfrak{h}_n + \tfrac{|w_n(x)|}{2} \right) \\
&= \left(\tfrac{4n+1}{4n(n+1)}, \tfrac{4n+3}{4n(n+1)} \right)
\end{aligned}
$$

so that

$$
\begin{aligned}
(89) \qquad \ell w_n \quad &= |w_n(x)| \\
&= \int_0^1 n(xn + x - 1)\chi(x, I_n^H)\mathrm{d}x \\
&= \int_{\frac{1}{n+1}}^{\frac{1}{n}} w(x)\mathrm{d}x \\
&= |I_n^w| \\
&= \int_0^1 \chi(x, I_n^w)\mathrm{d}x \\
&= \tfrac{4n+3}{4n(n+1)} - \tfrac{4n+1}{4n(n+1)} \\
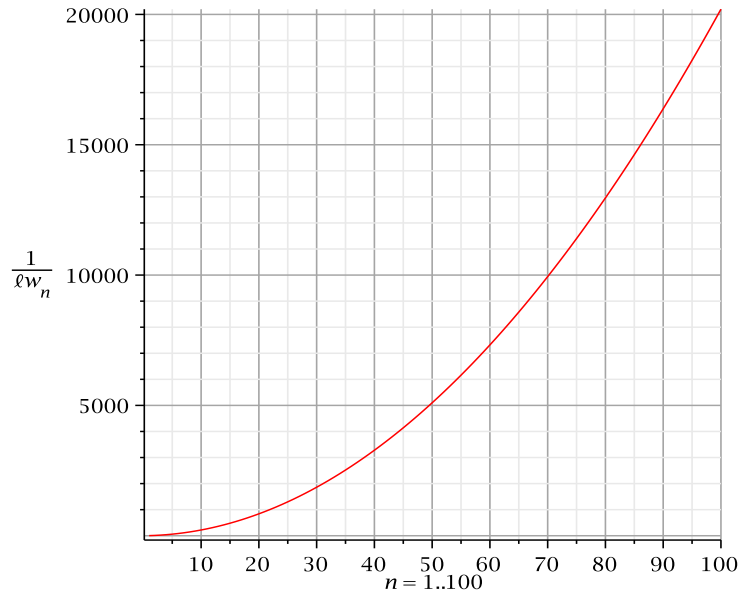&= \tfrac{1}{2n(n+1)}
\end{aligned}
$$

FIGURE 10. Reciprocal lengths ℓw_n^{-1}

The total length of $\mathcal{L}_w$ is

$$
\begin{aligned}
(90) \qquad |\mathcal{L}_w| \quad &= \int_0^1 w(x)\mathrm{d}x \\
&= \sum_{n=1}^{\infty} \ell w_n \\
&= \sum_{n=1}^{\infty} \tfrac{1}{2n(n+1)} \\
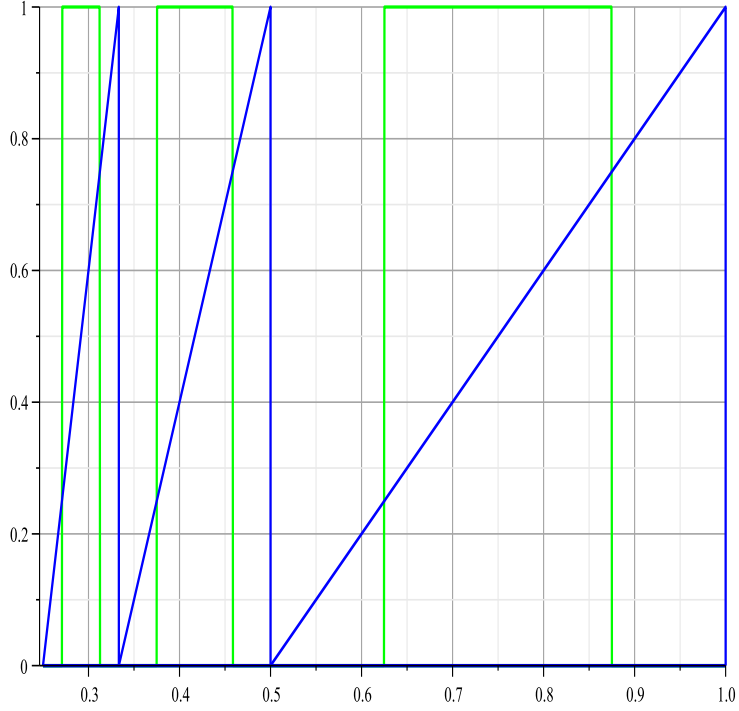&= \tfrac{1}{2}
\end{aligned}
$$

FIGURE 11. $\chi(x, I_n^w)$ green and $w_n(x)$ blue for $n = 1 \ldots 3$ and $x = \frac{1}{4}..1$

1.4.2. *Geometry and Volume of the Inner Tubular Neighborhood.* The geometric counting function (109) of $\mathcal{L}_w$ is

$$
(91) \qquad \begin{aligned}
N_{\mathcal{L}_w}(x) &= \#\{n \geqslant 1 : \ell w_n^{-1} \leqslant x\} \\
&= \#\{n \geqslant 1 : 2(n+1)n \leqslant x\} \\
&= \left\lfloor \frac{\sqrt{2x+1}}{2} - \frac{1}{2} \right\rfloor
\end{aligned}
$$

which is used to calculate the limiting constant (111) C_w appearing in the equation for the Minkowski content

$$
(92) \qquad \begin{aligned}
C_w &= \lim_{x \to \infty} \frac{N_{\mathcal{L}_w}(x)}{x^{D_{\mathcal{L}_w}}} \\
&= \lim_{x \to \infty} \frac{\frac{\sqrt{2x+1}}{2} - \frac{1}{2}}{\sqrt{x}} \\
&= \frac{\sqrt{2}}{2}
\end{aligned}
$$

The function $N_{\mathcal{L}_w}(x)$ happens to coincide with [17, A095861], which is the number of primitive Pythagorean triangles of the form $\{(a, b, b+1) : (b+1) \leqslant n\}$. [6, 171-176][37, 10.1][18, 11.2-11.5] Let

$$
(93) \qquad v(\varepsilon) = \min(j : \ell w_j < 2\varepsilon) = \left\lfloor \frac{\varepsilon + \sqrt{\varepsilon^2 + \varepsilon}}{2\varepsilon} \right\rfloor
$$

which is the floor of the solution to the inverse length equation

$$(94) \quad \begin{aligned} \frac{\varepsilon + \sqrt{\varepsilon^2 + \varepsilon}}{2\varepsilon} &= \{n : \ell w_{n-1} = 2\varepsilon\} = \left\{ n : \frac{1}{2n(n-1)} = 2\varepsilon \right\} \\ \frac{\ell w_{\frac{\varepsilon + \sqrt{\varepsilon^2 + \varepsilon}}{2\varepsilon} - 1}}{2} &= \varepsilon \end{aligned}$$

Then the volume of the inner tubular neighborhood of $\partial \mathcal{L}_w$ with radius ε (108) is

$$(95) \quad \begin{aligned} V_{\mathcal{L}_w}(\varepsilon) &= 2\varepsilon N_{\mathcal{L}_w}\left(\frac{1}{2\varepsilon}\right) + \sum_{j}^{\ell w_j < 2\varepsilon} \ell w_j \\ &= 2\varepsilon N_{\mathcal{L}_w}\left(\frac{1}{2\varepsilon}\right) + \sum_{n=v(\varepsilon)}^{\infty} \frac{1}{2(n+1)n} \\ &= 2\varepsilon N_{\mathcal{L}_w}\left(\frac{1}{2\varepsilon}\right) + \frac{1}{2v(\varepsilon)} \\ &= 2\varepsilon \left\lfloor \frac{\sqrt{\frac{1}{\varepsilon}+1}}{2} - \frac{1}{2} \right\rfloor + \frac{1}{2v(\varepsilon)} \\ &= \frac{4\varepsilon v(\varepsilon)^2 - 4\varepsilon v(\varepsilon) + 1}{2v(\varepsilon)} \end{aligned}$$

since

$$(96) \quad \sum_{n=m}^{\infty} \frac{1}{2n(n+1)} = \frac{1}{2m}$$

and by defintion we have

$$(97) \quad \begin{aligned} \lim_{\varepsilon \to 0^+} V_{\mathcal{L}_w}(\varepsilon) &= 0 \\ \lim_{\varepsilon \to \infty} V_{\mathcal{L}_w}(\varepsilon) &= |\mathcal{L}_w| = \frac{1}{2} \end{aligned}$$

Thus, using (92) and (95), the Minkowski content (110) of $\mathcal{L}_w$ is

$$(98) \quad \begin{aligned} \mathcal{M}_{\mathcal{L}_w} &= \lim_{e \to 0^+} \frac{V_{\mathcal{L}_w}(\varepsilon)}{\varepsilon^{1-D_{\mathcal{L}_w}}} \\ &= \lim_{e \to 0^+} \frac{1}{\sqrt{\varepsilon}} \left(2\varepsilon \left\lfloor \frac{\sqrt{\frac{1}{\varepsilon}+1}}{2} - \frac{1}{2} \right\rfloor + \frac{1}{2} \left\lfloor \frac{\varepsilon + \sqrt{\varepsilon^2 + \varepsilon}}{2\varepsilon} \right\rfloor^{-1} \right) \\ &= \frac{C_w 2^{1-D_{\mathcal{L}_w}}}{1 - D_{\mathcal{L}_w}} \\ &= \frac{\frac{\sqrt{2}}{2} 2^{1-\frac{1}{2}}}{1 - \frac{1}{2}} \\ &= 2 \end{aligned}$$

1.4.3. *The Geometric Zeta Function $\zeta_{\mathcal{L}_w}(s)$.* The geometric zeta function (112) of $w(x)$ is the Dirichlet series of the lengths ℓw_n (89) and also an integral over the geometric length counting function (91) $N_{\mathcal{L}_w}(x)$

$$(99) \quad \begin{aligned} \zeta_{\mathcal{L}_w}(s) &= \sum_{n=1}^{\infty} \ell w_j^s \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{2n(n+1)} \right)^s \\ &= \sum_{n=1}^{\infty} 2^{-s}(n+1)^{-s} n^{-s} \\ &= s \int_0^{\infty} N_{\mathcal{L}_w}(x) x^{-s-1} \mathrm{d}x \\ &= s \int_0^{\infty} \left\lfloor \frac{\sqrt{2x+1}}{2} - \frac{1}{2} \right\rfloor x^{-s-1} \mathrm{d}x \end{aligned}$$
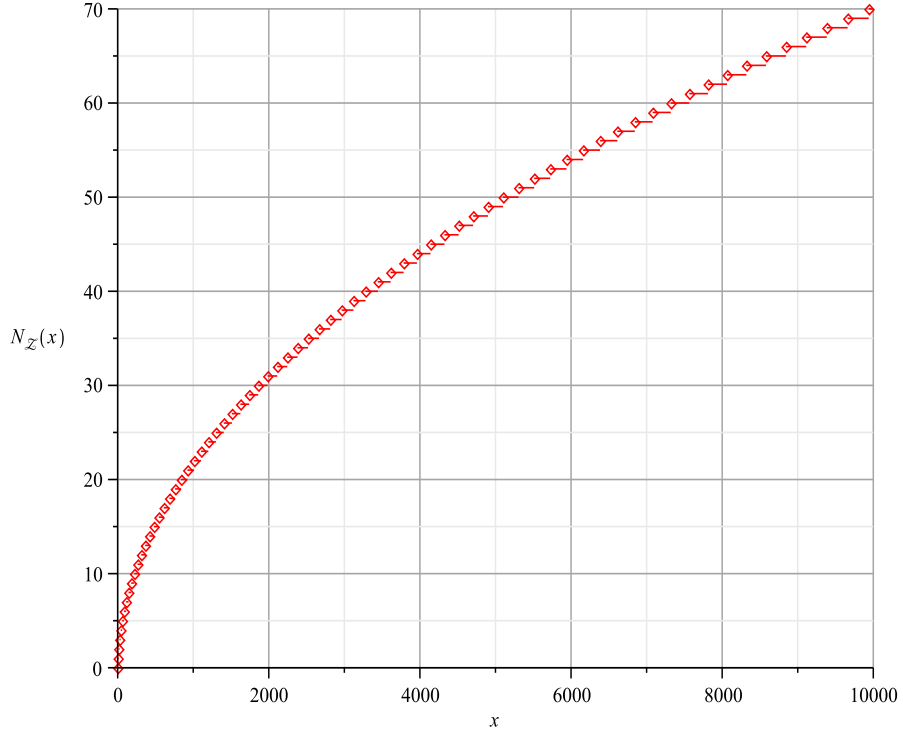
FIGURE 12. Geometric Counting Function $N_{\mathcal{L}_w}(x)$ of $w(x)$

The residue (115) of $\zeta_{\mathcal{L}_w}(s)$ at $D_{\mathcal{L}_w}$ is

$$
\begin{aligned}
\operatorname*{Res}_{s=D_{\mathcal{L}}}(\zeta_{\mathcal{L}}(s)) \quad &= \lim_{s\to D_{\mathcal{L}_w}^+} (s - D_{\mathcal{L}})\zeta_{\mathcal{L}}(s) \\
&= \lim_{s\to \frac{1}{2}^+} \left(s - \tfrac{1}{2}\right) \zeta_{\mathcal{L}}(s) \\
(100) \qquad &= \lim_{s\to \frac{1}{2}^+} \left(s - \tfrac{1}{2}\right) \sum_{n=1}^{\infty} 2^{-s}(n+1)^{-s}n^{-s} \\
&= \lim_{s\to \frac{1}{2}^+} \left(s - \tfrac{1}{2}\right) s \int_0^{\infty} \left\lfloor \tfrac{\sqrt{2x+1}}{2} - \tfrac{1}{2} \right\rfloor x^{-s-1}\mathrm{d}x \\
&= 0
\end{aligned}
$$

The values of $\zeta_{\mathcal{L}_w}(n)$ at positive integer values $n \in \mathbb{N}^*$ are given explicitly by a rather unwieldy sum of binomial coefficients and the Riemann zeta function $\zeta(n)$ at even integer values. First, define

$$
\begin{aligned}
(101) \qquad
a_n &= \tfrac{(n-1)\left(1-(-1)^{n+1}\right)}{2} \\
b_n &= \tfrac{(-1)^{n+1}(n-1)}{2} + n - \tfrac{7}{4} + \tfrac{(-1)^n}{4} \\
c_n &= (-1)^n(n-1) \\
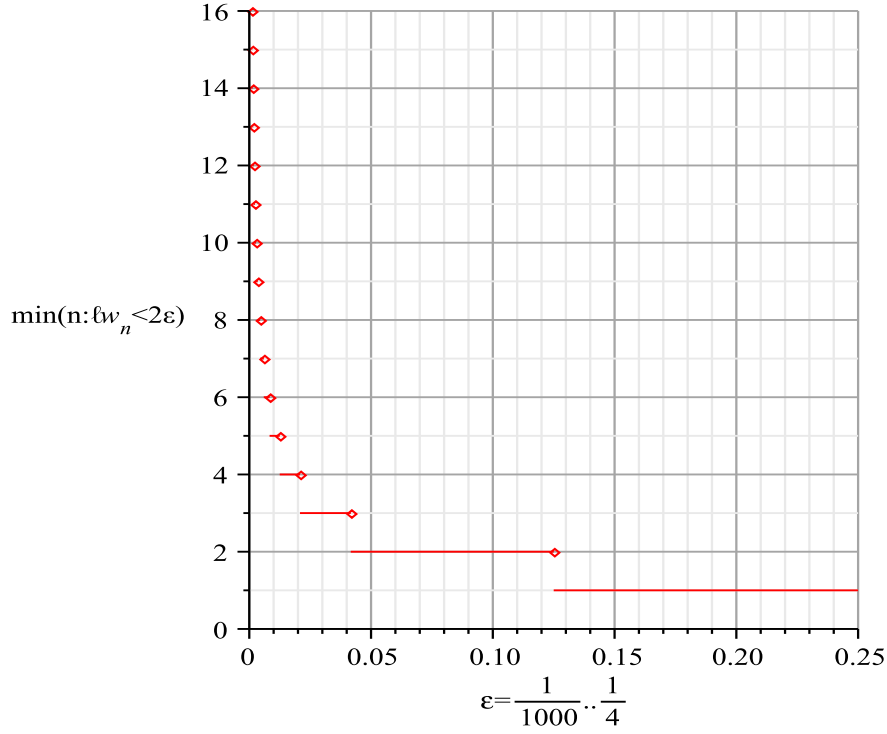d_n &= \tfrac{(-1)^n}{2}
\end{aligned}
$$

then

FIGURE 13. $\{v(\varepsilon) = \min(n : \ell w_n < 2\varepsilon) : \varepsilon = \frac{1}{1000} \cdots \frac{1}{4}\}$

$$(102) \qquad \zeta_{\mathcal{L}_w}(n) = \frac{(-1)^n \binom{2n-1}{n-1}}{2^n} + \sum_{m=a_n}^{b_n} \frac{2(-1)^n \binom{2m+c_n-d_n+\frac{1}{2}}{n-1}\zeta\left(d_n+2n-\frac{3}{2}-2m-c_n\right)}{2^n}$$

The terms of $\zeta_{\mathcal{L}_w}(n)$ from $n = 1$ to 10 are shown below in Table 1.

$$\begin{pmatrix}
+\frac{1}{2} \\
-\frac{3}{4} & +\frac{1}{2}\,\zeta(2) \\
+\frac{5}{4} & -\frac{3}{4}\,\zeta(2) \\
-\frac{35}{16} & +\frac{5}{4}\,\zeta(2) & +\frac{1}{8}\,\zeta(4) \\
+\frac{63}{16} & -\frac{35}{16}\,\zeta(2) & -\frac{5}{16}\,\zeta(4) \\
-\frac{231}{32} & +\frac{63}{16}\,\zeta(2) & +\frac{21}{32}\,\zeta(4) & +\frac{1}{32}\zeta(6) \\
+\frac{429}{32} & -\frac{231}{32}\,\zeta(2) & -\frac{21}{16}\,\zeta(4) & -\frac{7}{64}\,\zeta(6) \\
-\frac{6435}{256} & +\frac{429}{32}\,\zeta(2) & +\frac{165}{64}\,\zeta(4) & +\frac{9}{32}\,\zeta(6) & +\frac{1}{128}\,\zeta(8) \\
+\frac{12155}{256} & -\frac{6435}{256}\,\zeta(2) & -\frac{1287}{256}\,\zeta(4) & -\frac{165}{256}\,\zeta(6) & -\frac{9}{256}\,\zeta(8) \\
-\frac{46189}{512} & +\frac{12155}{256}\,\zeta(2) & +\frac{5005}{512}\,\zeta(4) & +\frac{715}{512}\,\zeta(6) & +\frac{55}{512}\,\zeta(8)+ & \frac{1}{512}\,\zeta(10)
\end{pmatrix}$$

TABLE 1. $\{\zeta_{\mathcal{L}_w}(n) = \Sigma\,\mathrm{row}_n : \text{n=1..10}\}$
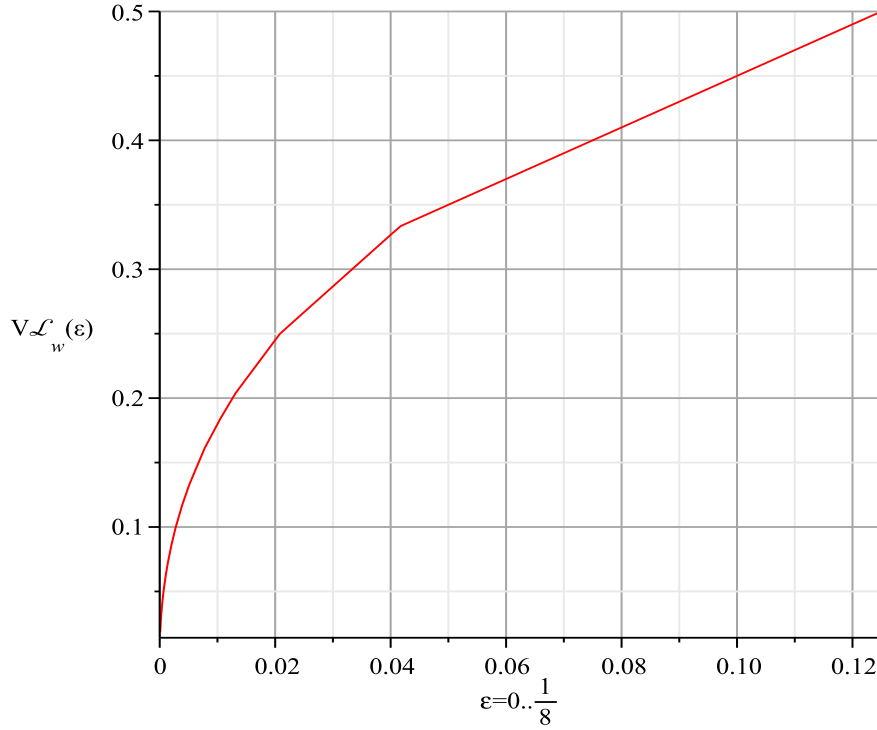
FIGURE 14. Volume of the inner tubular neighborhood of $\partial \mathcal{L}_w$ with radius ε $\left\{ V_{\mathcal{L}_w}(\varepsilon) : \varepsilon = 0 \ldots \frac{1}{8} \right\}$

2. FRACTAL STRINGS AND DYNAMICAL ZETA FUNCTIONS

2.1. **Fractal Strings.** A a fractal string $\mathcal{L}$ is defined as a nonempty bounded open subset of the real line $\mathcal{L} \subseteq \mathbb{R}$ consisting of a countable disjoint union of open intervals I_j

$$(103) \qquad \mathcal{L} = \bigcup_{j=1}^{\infty} I_j$$

The length of the j-th interval I_j is denoted by

$$(104) \qquad \ell_j = |I_j|$$

where $|\cdot|$ is the 1-dimensional Lebesgue measure. The lengths ℓ_j must form a nonnegative monotically decreasing sequence and the total length must be finite, that is

$$(105) \qquad \begin{array}{l} |\mathcal{L}|_1 = \sum_{j=1}^{\infty} \ell_j < \infty \\ \ell_1 \geqslant \ell_2 \geqslant \ldots \geqslant \ell_j \geqslant \ell_{j+1} \geqslant \cdots \geqslant 0 \end{array}$$

The case when $\ell_j = 0$ for any j will be excluded here since ℓ_j is a finite sequence. The fractal string is defined completely by its sequence of lengths so it can be denoted

$$(106) \qquad \mathcal{L} = \{\ell_j\}_{j=1}^{\infty}$$
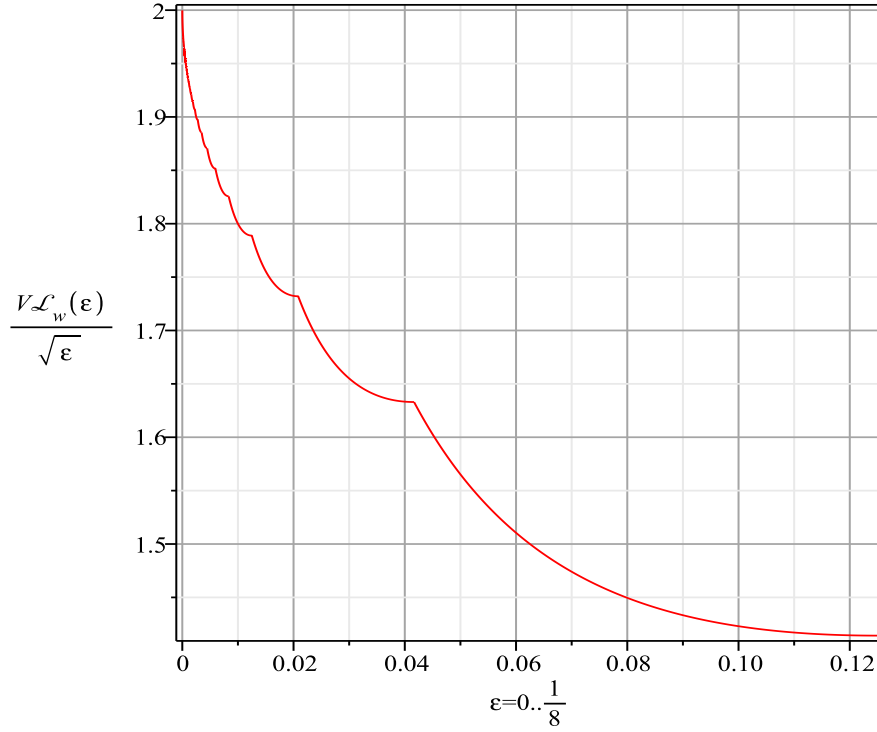
FIGURE 15. $\left\{ \frac{V\mathcal{L}_w(\varepsilon)}{\sqrt{\varepsilon}} : \varepsilon = 0 \dots \frac{1}{8} \right\}$ and $\frac{V\mathcal{L}_w(8^{-1})}{\sqrt{8^{-1}}} = \sqrt{2}$

The boundary of $\mathcal{L}$ in $\mathbb{R}$, denoted by $\partial\mathcal{L} \subset \Omega$, is a totally disconnected bounded perfect subset which can be represented as a string of finite length, and generally any compact subset of $\mathbb{R}$ also has this property. The boundary $\partial\mathcal{L}$ is said to be perfect since it is closed and each of its points is a limit point. Since the Cantor-Bendixon lemma states that there exists a perfect set $P \subset \partial\mathcal{L}$ such that $\partial\mathcal{L} - P$ is a most countable, we can define $\mathcal{L}$ as the complement of $\partial\mathcal{L}$ in its closed convex hull. The connected components of the bounded open set $\mathcal{L}\backslash\partial\mathcal{L}$ are the intervals I_j. [25, 1.2][32, 2.2 Ex17][23, 3.1][30][21][20][15][22][11][19][29]

2.1.1. *The Minkowski Dimension $D_{\mathcal{L}}$ and Content $\mathcal{M}_{\mathcal{L}}$.* The Minkowski dimension $D_{\mathcal{L}} \in [0,1]$, also known as the box dimension, is maximum value of $V(\varepsilon)$

(107) $$D_{\mathcal{L}} \quad = \inf\{\alpha \geqslant 0 : V(\varepsilon) = O(\varepsilon^{1-\alpha}) \text{ as } \varepsilon \to 0^+\} = \zeta_{\mathcal{L}}(1) = \sum_{n=1}^{\infty} \ell_j$$

where $V(\varepsilon)$ is the volume of the inner tubular neighborhoods of $\partial\mathcal{L}$ with radius ε

(108)
$$\begin{aligned} V_{\mathcal{L}}(\varepsilon) \quad &= |x \in \mathcal{L} : d(x,\partial\mathcal{L}) < \varepsilon| \\ &= \sum_j^{\ell_j \geqslant 2\varepsilon} 2\varepsilon + \sum_j^{\ell_j < 2\varepsilon} \ell_j \\ &= 2\varepsilon N_{\mathcal{L}}\left(\frac{1}{2\varepsilon}\right) + \sum_j^{\ell_j < 2\varepsilon} \ell_j \end{aligned}$$

and $N_{\mathcal{L}}(x)$ is the geometric counting function which is the number of components with their reciprocal length being less than or equal to x.

(109)
$$\begin{aligned} N_{\mathcal{L}}(x) \quad &= \#\{j \geqslant 1 : \ell_j^{-1} \leqslant x\} \\ &= \sum_{j \geqslant 1}^{\ell_j^{-1} \leqslant x} 1 \end{aligned}$$

The Minkowski content of $\mathcal{L}$ is then defined as

(110)
$$\begin{aligned} \mathcal{M}_{\mathcal{L}} \quad &= \lim_{e \to 0^+} \frac{V_{\mathcal{L}}(\varepsilon)}{\varepsilon^{1-D_{\mathcal{L}}}} \\ &= \frac{C_{\mathcal{L}} 2^{1-D_{\mathcal{L}}}}{1-D_{\mathcal{L}}} \\ &= \frac{\mathrm{Res}(\zeta_{\mathcal{L}}(s);D_{\mathcal{L}})2^{1-D_{\mathcal{L}}}}{D_{\mathcal{L}}(1-D_{\mathcal{L}})} \end{aligned}$$

where $C_{\mathcal{L}}$ is the constant

(111)
$$C_{\mathcal{L}} = \lim_{x \to \infty} \frac{N_{\mathcal{L}}(x)}{x^{D_{\mathcal{L}}}}$$

If $\mathcal{M}_{\mathcal{L}} \in (0,\infty)$ exists then $\mathcal{L}$ is said to be Minkowski measurable which necessarily means that the geometry of $\mathcal{L}$ does not oscillate and vice versa. [27, 1] [3][24][26, 6.2]

2.1.2. *The Geometric Zeta Function $\zeta_{\mathcal{L}}(s)$.* The geometric Zeta function $\zeta_{\mathcal{L}}(s)$ of $\mathcal{L}$ is the Dirichlet series

(112)
$$\begin{aligned} \zeta_{\mathcal{L}}(s) \quad &= \sum_{n=1}^{\infty} \ell_j^s \\ &= s \int_0^{\infty} N_{\mathcal{L}_w}(x) x^{-s-1} \mathrm{d}x \end{aligned}$$

which is holomorphic for $\mathrm{Re}(s) > D_{\mathcal{L}}$. If $\mathcal{L}$ is Minkowski measurable then $0 < D_{\mathcal{L}} < 1$ is the simple unique pole of $\zeta_{\mathcal{L}}(s)$ on the vertical line $\mathrm{Re}(s) = D_{\mathcal{L}}$. Assuming $\zeta_{\mathcal{L}}(s)$ has a meromorphic extension to a neighboorhood of $D_{\mathcal{L}}$ then $\zeta_{\mathcal{L}}(s)$ has a simple pole at $\zeta_{\mathcal{L}}(D_{\mathcal{L}})$ if

(113)
$$N_{\mathcal{L}}(s) = O(s^{D_{\mathcal{L}}}) \text{ as } s \to \infty$$

or if the volume of the tubular neighborhoods satisfies

(114)
$$V_{\mathcal{L}}(\varepsilon) = O(\varepsilon^{1-D_{\mathcal{L}}}) \text{ as } \varepsilon \to 0^+$$

It can be possible that the residue of $\zeta_{\mathcal{L}}(s)$ at $s = D_{\mathcal{L}}$ is positive and finite

(115)
$$0 < \lim_{s \to D_{\mathcal{L}}} (s - D_{\mathcal{L}})\zeta_{\mathcal{L}}(s) < \infty$$

even if $N_{\mathcal{L}}(s)$ is not of order $s^{D_{\mathcal{L}}}$ as $s \to \infty$ and $V_{\mathcal{L}}(\varepsilon)$ is not of order $\varepsilon^{1-D_{\mathcal{L}}}$, however this does not contradict the Minkowski measurability of $\mathcal{L}$.

2.1.3. *Complex Dimensions, Screens and Windows.* The set of visible complex dimensions of $\mathcal{L}$, denoted by $\mathcal{D}_\mathcal{L}(W)$, is a discrete subset of $\mathbb{C}$ consisting of the poles of $\{\zeta_\mathcal{L}(s) : s \in W\}$.

$$(116) \qquad \mathcal{D}_\mathcal{L}(W) = \{w \in W : \zeta_\mathcal{L}(w) \text{ is a pole}\}$$

When W is the entire complex plane then the set $\mathcal{D}_\mathcal{L}(\mathbb{C}) = \mathcal{D}_\mathcal{L}$ is simply called the set of complex dimensions of $\mathcal{L}$. The presence of oscillations in $V(\varepsilon)$ implies the presence of imaginary complex dimensions with $\mathrm{Re}(\cdot) = D_\mathcal{L}$ and vice versa. More generally, the complex dimensions of a fractal string $\mathcal{L}$ describe its geometric and spectral oscillations.

2.1.4. *Frequencies of Fractal Strings and Spectral Zeta Functions.* The eigenvalues λ_n of the Dirichlet Laplacian $\Delta u(x) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} u(x)$ on a bounded open set $\Omega \subset \mathbb{R}$ correspond to the normalized frequencies $f_n = \frac{\sqrt{\lambda_n}}{\pi}$ of a fractal string. The frequencies of the unit interval are the natural numbers $n \in \mathbb{N}^*$ and the frequencies of an interval of length ℓ are $n\ell^{-1}$. The frequencies of $\mathcal{L}$ are the numbers

$$(117) \qquad f_{k,j} = k\ell_j^{-1} \forall\, k, j \in \mathbb{N}^*$$

The spectral counting function $N_{v\mathcal{L}}(x)$ counts the frequencies of $\mathcal{L}$ with multiplicity

$$(118) \qquad \begin{aligned} N_{v\mathcal{L}}(x) &= \sum_{j=1}^\infty N_\mathcal{L}\left(\frac{x}{j}\right) \\ &= \sum_{j=1}^\infty \lfloor x\ell_j \rfloor \end{aligned}$$

The spectral zeta function $\zeta_{v\mathcal{L}}(s)$ of $\mathcal{L}$ is connected to the Riemann zeta function (**??**) by

$$(119) \qquad \begin{aligned} \zeta_{v\mathcal{L}}(s) &= \sum_{k=1}^\infty \sum_{j=1}^\infty k^{-s}\ell_j^s \\ &= \zeta(s)\sum_{n=1}^\infty \ell_j^s \\ &= \zeta(s)\zeta_\mathcal{L}(s) \end{aligned}$$

[27, 1.1][25, 1.2.1]

2.1.5. *Generalized Fractal Strings and Dirichlet Integrals.* A generalized fractal string is a positive or complex or local measure $\eta(x)$ on $(0,\infty)$ such that

$$(120) \qquad \int_0^{x_0} \eta(x)\mathrm{d}x = 0$$

for some $x_0 > 0$. A local positive measure is a standard positive Borel measure $\eta(J)$ on $(0,\infty)$ where J is the set of all bounded subintervals of $(0,\infty)$ in which case $\eta(x) = |\eta(x)|$. More generally, a meausre $\eta(x)$ is a local complex measure if $\eta(A)$ is well-defined for any subset $A \subset [a,b]$ where $[a,b] \subset [0,\infty]$ is a bounded subset of the positive half-line $(0,\infty)$ and the restriction of η to the Borel subsets of $[a,b]$ is a complex measure on $[a,b]$ in the traditional sense. The geometric counting function of $\eta(x)$ is defined as

$$(121) \qquad N_\eta(x) = \int_0^x \eta(x)\mathrm{d}x$$

The dimension D_η is the abscissa of conergence of the Dirichlet integral

$$(122) \qquad \zeta_{|\eta|}(\sigma) = \int_0^\infty x^{-\sigma}|\eta(x)|\mathrm{d}x$$

In other terms, it is the smallest real positive σ such that the improper Riemann-Lebesgue converges to a finite value. The geometric zeta function is defined as the Mellin transform

$$(123) \qquad \zeta_\eta(s) = \int_0^\infty x^{-s}\eta(x)\mathrm{d}x$$

where $\mathrm{Re}(s) > D_\eta$.

2.2. Fractal Membranes and Spectral Partitions.

2.2.1. *Complex Dimensions of Dynamical Zeta Functions.* The fractal membrane $\mathcal{T}_L$ associated with $\mathcal{L}$ is the adelic product

$$(124) \qquad \mathcal{T}_L = \coprod_{j=1}^{\infty} \mathcal{T}_j$$

where each $\mathcal{T}_j$ is an interval I_j of length $\log(\ell_j^{-1})^{-1}$. To each $\mathcal{T}_j$ is associated a Hilbert space $\mathcal{H}_j = L^2(I_j)$ of square integrable functions on I_j. The spectral partition function $Z_{\mathcal{L}}(s)$ of $\mathcal{L}$ is a Euler product expansion which has no zeros or poles in $\mathrm{Re}(s) > D_M(\mathcal{L})$.

$$(125) \qquad \begin{aligned} Z_{\mathcal{L}}(s) \quad &= \textstyle\prod_{j=1}^{\infty} \frac{1}{1-\ell_j^s} \\ &= \textstyle\prod_{j=1}^{\infty} Z_{\mathcal{L}_j}(s) \end{aligned}$$

where $D_M(\mathcal{L})$ is the Minkowski dimension of $\mathcal{L}$ and $Z_{\mathcal{L}_j}(s) = \frac{1}{1-\ell_j^s}$ is the j-th Euler factor, the partition function of the j-th component of the fractal membrane. [23, 3.2.2]

2.2.2. *Dynamical Zeta Functions of Fractal Membranes.* The dynamical zeta function of a fractal membrane $\mathcal{L}$ is the negative of the logarithmic derivative of the Zeta function associated with $\mathcal{L}$.

$$(126) \qquad \begin{aligned} Z_{\mathcal{L}}(s) \quad &= -\frac{\mathrm{d}}{\mathrm{d}s} \ln(\zeta_{\mathcal{L}}(s)) \\ &= -\frac{\frac{\mathrm{d}}{\mathrm{d}s}\zeta_{\mathcal{L}}(s)}{\zeta_{\mathcal{L}}(s)} \end{aligned}$$

3. Special Functions, Definitions, and Conventions

3.1. Special Functions.

3.1.1. *The Interval Indicator (Characteristic) Function $\chi(x, I)$.* The (left-open, right-closed) interval indicator function is $\chi(x, I)$ where $I = (a, b]$

$$(127) \qquad \begin{aligned} \chi(x, I) \quad &= \begin{cases} 1 & x \in I \\ 0 & x \notin I \end{cases} \\ &= \begin{cases} 1 & a < x \leqslant b \\ 0 & \text{otherwise} \end{cases} \\ &= \theta(x-a) - \theta(x-a)\theta(x-b) \end{aligned}$$

and θ is the Heaviside unit step function, the derivative of which is the Dirac delta function δ

$$(128) \qquad \begin{aligned} \int \delta(x)\mathrm{d}x \quad &= \theta(x) \\ \theta(x) \quad &= \begin{cases} 0 & x < 0 \\ 1 & x \geqslant 0 \end{cases} \end{aligned}$$

The discontinous point of $\theta(x)$ has the limiting values

$$(129) \qquad \begin{aligned} \lim_{x \to 0^-} \theta(x) \quad &= 0 \\ \lim_{x \to 0^+} \theta(x) \quad &= 1 \end{aligned}$$

thus the values of $\chi(x, (a, b))$ on the boundary can be chosen according to which side the limit is regarded as being approached from.

$$(130) \qquad \begin{aligned} \lim_{x \to a^-} \chi(x, (a, b]) \quad &= 0 \\ \lim_{x \to a^+} \chi(x, (a, b]) \quad &= 1 - \theta(a - b) \\ \lim_{x \to b^-} \chi(x, (a, b]) \quad &= \theta(b - a) \\ \lim_{x \to b^+} \chi(x, (a, b]) \quad &= 0 \end{aligned}$$

3.1.2. *"Harmonic" Intervals.* Let the n-th harmonic (left-open, right-closed) interval be defined as

$$(131) \qquad I_n^H = \left(\tfrac{1}{n+1}, \tfrac{1}{n} \right]$$

then its characteristic function is

$$(132) \qquad \begin{aligned} \chi(x, I_n^H) &= \theta\left(x - \tfrac{1}{n+1}\right) - \theta\left(x - \tfrac{1}{n}\right) \\ &= \theta\left(x - \tfrac{n}{n+1}\right) - \theta\left(x - n\right) \\ &= \begin{cases} 1 & \tfrac{1}{n+1} < x \leqslant \tfrac{1}{n} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

As can be seen

$$(133) \qquad \begin{aligned} \bigcup_{n=1}^{\infty} I_n^H &= \bigcup_{n=1}^{\infty} \left(\tfrac{1}{n+1}, \tfrac{1}{n} \right] &= (0, 1] \\ \sum_{n=1}^{\infty} \chi(x, I_n^H) &= \sum_{n=1}^{\infty} \chi\left(x, \left(\tfrac{1}{n+1}, \tfrac{1}{n} \right]\right) &= \chi(x, (0, 1]) \end{aligned}$$

The substitution $n \to \lfloor \tfrac{1}{x} \rfloor$ can be made in (132) where it is seen that

$$(134) \qquad \chi\left(x, I_{\lfloor x^{-1} \rfloor}^H\right) = \theta\left(\tfrac{x\lfloor x^{-1} \rfloor + x - 1}{\lfloor x^{-1} \rfloor + 1}\right) - \theta\left(\tfrac{x\lfloor x^{-1} \rfloor - 1}{\lfloor x^{-1} \rfloor}\right) = 1 \quad \forall x \in [-1, +1]$$

3.1.3. *The Laplace Transform $L_a^b[f(x); x \to s]$.* The Laplace transform [2, 1.5] is defined as

$$(135) \qquad L_a^b[f(x); x \to s] = \int_a^b f(x) e^{-xs} \mathrm{d}x$$

where the unilateral Laplace transform is over the interval $(a, b) = (0, \infty)$ and the bilateral transform is over $(a, b) = (-\infty, \infty)$. When (a, b) is not specified, it is assumed to range over the support of $f(x)$ if the support is an interval. If the support of $f(x)$ is not an interval then (a, b) must be specified. Applying L to the interval indicator function (127) gives

$$(136) \qquad \begin{aligned} L_a^b[\chi(x, (a, b)); x \to s] &= \int_a^b \chi(x, (a, b)) e^{-xs} \mathrm{d}x \\ &= \int_a^b (\theta(x - a) - \theta(x - b)\theta(x - a)) e^{-xs} \mathrm{d}x \\ &= \tfrac{e^{bs} - e^{as}}{se^{bs}e^{as}} \\ &= -\tfrac{(e^{as} - e^{bs})e^{-s(b+a)}}{s} \end{aligned}$$

The limit at the singular point $s = 0$ is

$$(137) \qquad \begin{aligned} \lim_{s \to 0} L_a^b[\chi(x, (a, b)); x \to s] &= \lim_{s \to 0} -\tfrac{(e^{as} - e^{bs})e^{-s(b+a)}}{s} \\ &= b - a \end{aligned}$$

3.1.4. *The Mellin Transform $M_a^b[f(x); x \to s]$.* The Mellin transform [31, 3.2][33, II.10.8][5, 3.6] is defined as

$$(138) \qquad M_a^b[f(x); x \to s] = \int_a^b f(x) x^{s-1} \mathrm{d}x$$

where the standard Mellin transform is over the interval $(a, b) = (0, \infty)$. Again, as with the notation for Laplace transform, the integral is over the support of $f(x)$ if the support is an interval and (a, b)

is not specified, otherwise (a, b) must be specified. Applying M to the interval indicator function (127) gives

$$
(139) \qquad
\begin{aligned}
M_a^b[\chi(x, (a, b)); x \to s] &= \int_a^b \chi(x, (a, b)) x^{s-1} \mathrm{d}x \\
&= \int_a^b (\theta(x - a) - \theta(x - b)\theta(x - a)) x^{s-1} \mathrm{d}x \\
&= \frac{b^s - a^s}{s}
\end{aligned}
$$

The limit at the singular point $s = 0$ is

$$
(140) \qquad
\begin{aligned}
M_a^b[\chi(x, (a, b)); x \to 0] &= M[\chi(x, (a, b)); x \to 0] \\
&= \lim_{s \to 0} M_a^b[\chi(x, (a, b)); x \to s] \\
&= \lim_{s \to 0} \frac{b^s - a^s}{s} \\
&= \ln(b) - \ln(a)
\end{aligned}
$$

The Mellin transform has several identities [31, 3.1.2], including but not limited to

$$
(141) \qquad
\begin{aligned}
M[f(\alpha x); x \to s] &= \alpha^{-s} M[f(x); x \to s] \\
M[x^\alpha f(x); x \to s] &= M[f(x); x \to s + \alpha] \\
M[f(x^\alpha); x \to s] &= \frac{1}{\alpha} M\left[f(x); x \to \frac{s}{\alpha}\right] \\
M[f(x^{-\alpha}); x \to s] &= \frac{1}{\alpha} M\left[f(x); x \to -\frac{s}{\alpha}\right] \\
M[x^\alpha f(x^\mu); x \to s] &= \frac{1}{\mu} M\left[f(x); x \to \frac{s+\alpha}{\mu}\right] \\[4pt]
M[x^\alpha f(x^{-\mu}); x \to s] &= \frac{1}{\mu} M\left[f(x); x \to -\frac{s+\alpha}{\mu}\right] \\
M[\ln(x)^n f(x); x \to s] &= \frac{\mathrm{d}^n}{\mathrm{d}s^n} M[f(x); x \to s]
\end{aligned}
$$

where $\alpha > 0, \mu > 0$, and $n \in \mathbb{N}$. The Mellin transform of the harmonic interval indicator function (132) is

$$
(142) \qquad
\begin{aligned}
M\left[\chi\left(x, I_n^H\right); x \to 0\right] &= \int_{\frac{1}{n+1}}^{\frac{1}{n}} \chi\left(x, \left(\frac{1}{n+1}, \frac{1}{n}\right)\right) x^{s-1} \mathrm{d}x \\
&= \int_{\frac{1}{n+1}}^{\frac{1}{n}} \left(\theta\left(t - \frac{1}{n+1}\right) - \theta\left(t - \frac{1}{n}\right)\right) x^{s-1} \mathrm{d}x \\
&= \frac{n^{-s} - (n+1)^{-s}}{s}
\end{aligned}
$$

which has the limit

$$
(143) \qquad
\begin{aligned}
M\left[\chi\left(x, I_n^H\right); x \to 0\right] &= \lim_{s \to 0} M\left[\chi\left(x, I_n^H\right); x \to s\right] \\
&= \lim_{s \to 0} \frac{n^{-s} - (n+1)^{-s}}{s} \\
&= \ln(n + 1) - \ln(n)
\end{aligned}
$$

The Mellin and bilaterial Laplace transforms are related by the change of variables $x \to -\ln(y)$ resulting in the identity [31, 3.1.1]

$$
(144) \qquad
\begin{aligned}
M_0^\infty[f(-\ln(x)); x \to s] &= L_{-\infty}^{+\infty}[f(y); y \to s] \\
\int_0^\infty f(-\ln(x)) x^{s-1} \mathrm{d}x &= \int_{-\infty}^{-\infty} f(y) e^{-ys} \mathrm{d}y
\end{aligned}
$$

3.1.5. *The Lambert W Function $W(k, x)$.* The Lambert W function [8][34] is the inverse of xe^x given by

$$
(145) \qquad
\begin{aligned}
W(z) &= \{x : xe^x = z\} \\
&= W(0, z) \\
&= 1 + (\ln(z) - 1) \exp\left(\frac{i}{2\pi} \int_0^\infty \frac{1}{x+1} \ln\left(\frac{x - i\pi - \ln(x) + \ln(z)}{x + i\pi - \ln(x) + \ln(z)}\right) \mathrm{d}x\right) \\
&= \sum_{k=1}^\infty \frac{(-k)^{k-1} z^k}{k!}
\end{aligned}
$$

where $W(a,z) \forall a \in \mathbb{Z}, z \notin \{0, -e^{-1}\}$ is

(146) $\qquad W(a,z) \quad = 1 + (2i\pi a + \ln(z) - 1)\exp\left(\frac{i}{2\pi}\int_0^\infty \frac{1}{x+1}\ln\left(\frac{x+(2\,a-1)i\pi-\ln(x)+\ln(z)}{x+(2\,a+1)i\pi-\ln(x)+\ln(z)}\right)dx\right)$

A generaliztion of (145) is solved by

(147) $\qquad\qquad\qquad\qquad\qquad \{x : xb^x = z\} \quad = \frac{W(\ln(b)z)}{\ln(b)}$

The W function satisifes several identities

(148) $\qquad\qquad
\begin{aligned}
W(z)e^{W(z)} \quad &= z \\
W(z\ln(z)) \quad &= \ln(z) \qquad\qquad\qquad &\forall z < 1 \\
|W(z)| \quad &= W(|z|) \\
e^{nW(z)} \quad &= z^n W(z)^{-n} \\
\ln(W(n,z)) \quad &= \ln(z) - W(n,z) + 2i\pi n \\
W\left(-\frac{\ln(z)}{z}\right) \quad &= -\ln(z) \qquad\qquad\qquad &\forall z \in [0,e] \\
\frac{W(-\ln(z))}{-\ln(z)} \quad &= z^{z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}}
\end{aligned}$

where $n \in \mathbb{Z}$. Some special values are

(149) $\qquad\qquad
\begin{aligned}
W\left(-1, -e^{-1}\right) \quad &= -1 \\
W(-e^{-1}) \quad &= -1 \\
W(e) \quad &= 1 \\
W(0) \quad &= 0 \\
W(\infty) \quad &= \infty \\
W(-\infty) \quad &= \infty + i\pi \\
W\left(-\frac{\pi}{2}\right) \quad &= \frac{i\pi}{2} \\
W\left(-\ln(\sqrt{2})\right) \quad &= -\ln(2) \\
W\left(-1, -\ln(\sqrt{2})\right) \quad &= -2\ln(2)
\end{aligned}$

We also have the limit

(150) $\qquad\qquad\qquad\qquad\qquad \lim_{a\to\pm\infty}\frac{W(a,x)}{a} \quad = 2\pi i$

and differential

(151) $\qquad\qquad\qquad \frac{\mathrm{d}}{\mathrm{d}z}W(a,f(z)) \qquad = \frac{W(a,f(z))\frac{\mathrm{d}}{\mathrm{d}z}f(z)}{f(z)(1+W(a,f(z)))}$

as well as the obvious integral

(152) $\qquad\qquad\qquad \int_0^1 W\left(-\frac{\ln(x)}{x}\right)\mathrm{d}x \quad = \int_0^1 -\ln(x)\mathrm{d}x = 1$

Let us define, for the sake of brevity, the function

(153) $\qquad
\begin{aligned}
W_{\ln}(z) \quad &= W\left(-1, -\frac{\ln(z)}{z}\right) \\
&= 1 + \left(\ln\left(-\frac{\ln(z)}{z}\right) - 1 - 2\pi i\right)\exp\left(\frac{i}{2\pi}\int_0^\infty \frac{1}{x+1}\ln\left(\frac{x-3i\pi-\ln(x)+\ln\left(-\frac{\ln(z)}{z}\right)}{x-i\pi-\ln(x)+\ln\left(-\frac{\ln(z)}{z}\right)}\right)dx\right)
\end{aligned}$

Then we have the limits

(154) $\qquad\qquad\qquad\qquad
\begin{aligned}
\lim_{x\to-\infty} W_{\ln}(x) \quad &= 0 \\
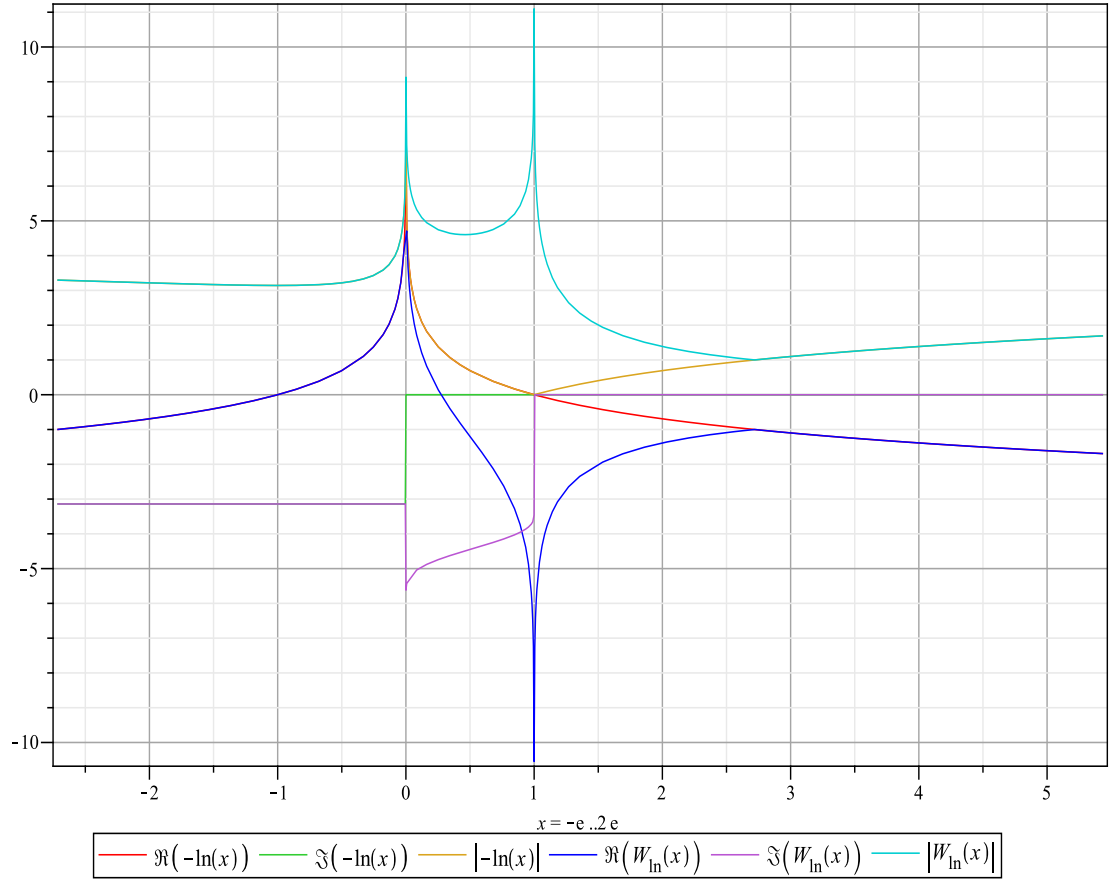\lim_{x\to+\infty} W_{\ln}(x) \quad &= -\infty
\end{aligned}$

FIGURE 16. $W\left(-\frac{\ln(x)}{x}\right) = -\ln(x)$ and $W_{\ln}(x) = W\left(-1, -\frac{\ln(x)}{x}\right)$

and

$$(155) \qquad \mathrm{Im}\left(W_{\ln}(x)\right) = \begin{cases} -\pi & -\infty < x < 0 \\ \ldots & 0 \leqslant x \leqslant 1 \\ 0 & 1 < x < \infty \end{cases}$$

$$(156) \qquad W_{\ln}(x) \quad = -\ln(x) \quad \forall x \notin [0, e]$$

The root of $\mathrm{Re}\left(W_{\ln}(x)\right)$ is given by

$$(157) \qquad \begin{aligned} \{x : \mathrm{Re}\left(W_{\ln}(x)\right) = 0\} \quad &= \{-x : \left(x^2\right)^{\frac{1}{x}} = e^{3\pi}\} \\ &= \tfrac{2}{3}\pi W\left(\tfrac{3}{2}\pi\right) \\ &\cong 0.27441063190284810044\ldots \end{aligned}$$

where the imaginary part of the value at the root of the real part of $W_{\ln}(z)$ is

(158)
$$
\begin{aligned}
W_{\ln}\left(\tfrac{2}{3}\pi W\left(\tfrac{3}{2}\pi\right)\right) &= W\left(-1, -\frac{\ln\left(\tfrac{2}{3}\pi W\left(\tfrac{3}{2}\pi\right)\right)}{\tfrac{2}{3}\pi W\left(\tfrac{3}{2}\pi\right)}\right) \\
&= W\left(-1, \tfrac{3\pi}{2}\right) \\
&= \tfrac{3\pi i}{2} \\
&\cong i4.712388980384689857\ldots
\end{aligned}
$$

3.1.6. *The Lerch Transcendent* $\Phi(z,a,v)$. The Lerch Transcendent [14, 1.11] is defined by

(159)
$$
\Phi(z,a,v) = \sum_{n=0}^{\infty} \frac{z^n}{(v+n)^a} \quad \forall\{|z|<1\}\,\text{or}\{|z|=1\,\text{and}\,\mathrm{Re}(a)>1\}
$$

The Riemann zeta function is the special case

(160)
$$
\zeta(s) = \Phi(1,s,1) = \sum_{n=0}^{\infty} \frac{1}{(1+n)^s}
$$

3.2. Applications of w(x).

3.2.1. *Expansion of* γ. Consider Euler's constant $\gamma = 0.577215664901533\ldots$ (33)

(161)
$$
w^n(\gamma) = a_n - b_n\gamma
$$

whereupon iteration we see that
(162)

$$
\begin{pmatrix} n \\ -a_n \\ -b_n \end{pmatrix} = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \ldots \\
0 & 1 & 48 & 290 & 581 & 1163 & 2327 & 13964 & 7492468716 & 14984937433 & 1078915495184 & \ldots \\
1 & 2 & 84 & 504 & 1008 & 2016 & 4032 & 24192 & 12980362752 & 25960725504 & 1869172236288 & \ldots
\end{pmatrix}
$$

3.3. **Conventions and Symbols.** Many of these symbols are from [23, p491].

(163)

i	$\sqrt{-1}$
$\mathbb{R}$	$\{x : -\infty < x < \infty\}$
$\bar{\mathbb{R}}$	$\{x : -\infty \leqslant x \leqslant \infty\}$
$\mathbb{R}^+$	$\{x : 0 \leqslant x < \infty\}$
$\mathbb{R}^d$	$\{x_1 \ldots x_d : -\infty < x_i < \infty\}$
$\mathbb{C}$	$\{x + iy : x, y \in \mathbb{R}\}$
$\mathbb{Z}$	$\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{N}$	$\{0, 1, 2, 3, \ldots\}$
$\mathbb{N}^*$	$\{1, 2, 3, \ldots\}$
$\mathbb{H}$	$\left\{0, \frac{1}{n} : n \in \mathbb{Z}\right\}$
$f(x) = O(g(x))$	$\frac{f(x)}{g(x)} < \infty$
$f(x) = o(g(x))$	$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$
$f(x) \asymp g(x)$	$\left\{a \leqslant \frac{f(x)}{g(x)} \leqslant b : \{a, b\} > 0\right\}$
$\#A$	numbers of elements in the finite set A
$|A|_d$	d-dimensional Lebesgue measure(volume) of $A \subseteq \mathbb{R}^d$
$d(x, A)$	$\{\min(|x - y|) : y \in A\}$ Euclidean distance between x and the nearest point of A
$\exp(x)$	exponential $e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$
$\operatorname{Res}_{x=y}(f(x))$	complex residue of $f(x)$ at $x = y$
$\lfloor x \rfloor$	floor, the greatest integer $\leqslant x$
$\{x\}$	$x - \lfloor x \rfloor$, the fractional part of x
$\bar{x}$	complex conjugate, $\operatorname{Re}(x) - \operatorname{Im}(x)$
Fix_f^n	n-th fixed point of the map $f(x)$, n-th solution to $f(x) = x$
p_k	k-th prime number
$\ln_b(a)$	$\frac{\ln(a)}{\ln(b)}$

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