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INFINITESIMAL GENERATOR OF MEAN ERGODIC THEOREM IN SEMIGROUP OF LINEAR OPERATOR

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Abstract. In this paper, results of Ergodic Theorem were obtained by investigating the behavior of ω-order reversing partial contraction mapping (semigroup of linear operator) as t tends to $+\infty$ of the means $\frac{1}{t}\int_0^t T(s)ds$ of a C_0-Semigroup of contraction $\{T(t); t \geq 0\}$ and it was shown that the infinitesimal generator of a C_0-Semigroup of contraction $\{T(t); t \geq 0\}$ is densely defined, linear, closed, reflexive and there exists a continuous bounded function.

1. Introduction

Ergodic theory can be described as the study of measurable maps, flows, and more generally group actions, preserving a certain measure of convergency on an operator. Let X be a Banach space, $X_n \subseteq X$ be a finite set, $(T(t))_{t\geq 0}$ the C_0-semigroup, $\omega - ORCP_n$ be ω-order-reversing partial contraction mapping which is an example of C_0-semigroup, $\omega - ORCP_n \subseteq ORCP_n$ (Order Reversing Partial Contraction Mapping). Let $Mm(\mathbb{N} \cup 0)$ be a matrix, $L(X)$ the bounded linear operator in X, P_n, the partial transformation semigroup, $\rho(A)$ a resolvent of A, where A is the generator of a semigroup of linear operator. This paper consist results on measure of convergence on invariant closed subspaces in Banach space. Ahmed [1], remarked some dynamics of impulsive systems in Banach spaces. Amann [2], investigated linear and quasilinear parabolic problems. Anita [3] established analysis and control of age-dependent population dynamics. Akinyele *et al.* [4], obtained some perturbation and linear delay equation results on ω-order reversing partial contraction mapping in semigroup of linear operator. Also in [5], Akinyele *et al.*, derived perturbation of infinitesimal generator in semigroup of linear operator. Barreira [6], obtained some results

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on ergodic theory, hyperbolic, dynamics and dimension theory. Bejenaru *et al.* [7], proved an abstract approximate controllability results and its application to elliptic and parabolic systems with dynamic boundary conditions. Engel and Nagel [8], obtained one-parameter semigroup for linear evolution equations. Feller [9], deduced parabolic differential equations and associated semigroups of transformation. Goldstein *et al.* [10], established convergence rates of ergodic limits for semigroups and cosine functions. Rauf and Akinyele [11], obtained ω-order-preserving partial contraction mapping and established its properties, also in [12], Rauf *et.al.* established some results of stability and spectra properties on semigroup of linear operator. Vrabie [13], characterized new generator of differentiable semigroups and also in [14], Vrabie deduced some results of C_0-semigroup and its applications. Widder [15], introduced and proved some theorems on laplace transforms. Yosida [16], established and proved some results on differentiability and representation of one-parameter semigroup of linear operators.

2. Preliminaries

Definition 2.1 (C_0-Semigroup) [14]
A C_0-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.
Definition 2.2 (Differentiable Semigroup) [14]
A C_0-Semigroup is a is called:
(i) differentiable at $\tau \geq 0$, if, for each $x \in X$, the function $t \to T(t)x$ is differentiable at τ;
(ii) differentiable, if it differentiable at each $\tau \in (0, +\infty)$; and
(iii) eventually differentiable if there exists $\theta > 0$ such that $t \to T(t)x$ is differentiable at each $\tau \in (\theta, +\infty)$.
Definition 2.3 (Ergodic Theory) [6]
Ergodic theory is a branch of mathematics that studies dynamical systems with an invariant measure and related problems.
Definition 2.4 (ω-$ORCP_n$) [11]
A transformation $\alpha \in P_n$ is called ω-order-reversing partial contraction mapping if $\forall x, y \in$Domα : $x \leq y \implies \alpha x \geq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.
Definition 2.5 (C_0-semigroup of contraction)[14]
A C_0-semigroup $\{T(t); t \geq 0\}$ is called of type (M, ω) with $M \geq 1$ and $\omega \in \mathbb{R}$, if for each $t \geq 0$, we have

$$\|T(t)\|_{L(X)} \leq Me^{t\omega}.$$

A C_0-semigroup $\{T(t); t \geq 0\}$ is called a C_0-semigroup of contraction or non expansive operator, if it is of type $(1, 0)$, that is, if for each $t \geq 0$, we have

$$\|T(t)\|_{L(X)} \leq 1.$$

Example 1
3×3 matrix $[M_m(\mathbb{C})]$, we have
for each $\lambda > 0$ such that $\lambda \in \rho(A)$ where $\rho(A)$ is a resolvent set on X.
Suppose we have

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA_\lambda}$, then

$$e^{tA_\lambda} = \begin{pmatrix} e^{3t\lambda} & e^{2t\lambda} & e^{t\lambda} \\ e^{2t\lambda} & e^{2t\lambda} & e^{t\lambda} \\ e^{3t\lambda} & e^{2t\lambda} & e^{2t\lambda} \end{pmatrix}.$$

Example 2
An operator $A : D(A) \subseteq X \to X$ is called closable if the closure of its graph is the graph of a linear operator $\bar{A}$, called the closure of the operator A and let $X = L^2(0, \pi)$ and defined $A : D(A) \subseteq X \to X$ by

$$\begin{cases} D(A) = \{u \in X; u^{'} \; there \; exists \; a.e. \; and \; u^{'} = va.e. \; with \; v \in X\} \\ Au = u' \; for \; each \; u \in D(A). \end{cases}$$

2.1. **Theorem.** Hille-Yoshida [12]
A linear operator $A : D(A) \subseteq X \to X$ is the infinitesimal generator for a C_0-semigroup of contraction if and only if

i. A is densely defined and closed,
ii. $(0, +\infty) \subseteq \rho(A)$ and for each $\lambda > 0$, we have

$$\|R(\lambda, A)\|_{L(X)} \leq \frac{1}{\lambda}. \tag{2.1}$$

3. Main Results

In this section, the mean ergodic theorem results on ω-$ORCP_n$ in semigroup of linear operator (C_0-semigroup) were established:

Theorem 3.1
Suppose $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup of contractions where $A \in \omega$-$ORCP_n$ and let $X_1 = D(A)$ endowed with the graph-norm $|.|_{D(A)} : X_1 \to \mathbb{R}_+$ defined by $|u|_{D(A)} = \|u - Au\|$ for $x \in X_1$. Then the operator $A_{(1)} : D(A_{(1)}) \subseteq X_1 \to X_1$ defined by

$$\begin{cases} D(A_{(1)}) = \{x \in X_1 : Ax \in X_1\} \\ A_{(1)}x = Ax, for \; all \; x \in D(A_{(1)}), \end{cases}$$

is the infinitesimal generator of a C_0-semigroup of contractions on X_1.
proof :
Suppose $\lambda > 0$, $f \in X$, $A \in \omega\text{-}ORCP_n$ and let us consider the equation

$$\lambda x - Au = f. \tag{3.1}$$

Since A generates a C_0-semigroup of contractions, by Theorem 2.1, its follows that this equation has a unique solution $u \in D(A)$. As $f \in X_1$, we deduced that $Au \in D(A)$, thus $u \in D(A_{(1)})$. Then,

$$\lambda u - A_{(1)}u = f. \tag{3.2}$$

On the other hand, we have

$$\begin{aligned} |(\lambda I - A_{(1)})f|_{D(A)} &= \|(I - A)(\lambda I - A)^{-1}f\| \\ . &= \|(\lambda I - A)^{-1}(I - A)f\| \leq \frac{1}{\lambda}\|f - Af\| = \frac{1}{\lambda}|f|_{D(A)} \end{aligned} \tag{3.3}$$

which shows that $A_{(1)}$ satisfies (ii) in Theorem 2.1. Moreover, it follows that $A_{(1)}$ is closed in X_1. Since $(\lambda I - A_{(1)})^{-1} \in L(X_1)$, then it is closed, and consequently $\lambda I - A_{(1)}$ enjoys the same property which proves that $A_{(1)}$ is closed too. Next, let $x \in X_1$, $\lambda > 0$, $A \in \omega\text{-}ORCP_n$ and let $x_\lambda = \lambda x - A_{(1)}x$. Clearly $x_1 \in D(A_{(1)})$, and in addition ,

$$\lim_{\lambda \to \infty} |x\lambda - x|_{D(A)} = 0, \tag{3.4}$$

and by virtue of $\lim_{\lambda \to \infty} A_\lambda x = Ax$, then (3.4) holds. Thus $D(A_{(1)})$ is dense in X_1 and by virtue of Theorem 2.1, $A_{(1)}$ generates a C_0-semigroup of contraction on X_1. Hence the proof.
Theorem 3.2
Let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup of contraction $\{T(t); t \leq 0\}$, such that $A \in \omega\text{-}ORCP_n$. Then
(i) if $x_1 \in ker(A)$ and $x_2 \in R(A)$, then there exists a bounded function $\varphi : \mathbb{R}_+ \to X$ with the property
$\lim_{t \to \infty} \frac{\varphi(t)}{t} = 0$, and such that, for each $t > 0$,
$\frac{1}{t}\int_0^t T(\tau)(x_1 + x_2)d\tau = x_1 + \frac{\varphi(t)}{t}$; and
(ii) if X is reflexive and $(A, D(A))$ is densely defined, linear, closed operator, we have
$ker(A^*)^\perp = \overline{R(A^*)}$ and $R(A)^\perp = ker A^*$.

Proof :
Let us observe first that, for each $x_1 \in ker(A)$ and each $\tau \geq 0$, we have $T(\tau)x_1 = x_1$. As $x_2 \in R(A)$, there exists $x_3 \in D(A)$ such that $x_2 = Ax_3$. Accordingly,

$$\frac{1}{t}\int_0^t T(\tau)(x_1 + x_2)d\tau = x_1 + \frac{1}{t}\int_0^t T(\tau)x_2 d\tau, \tag{3.5}$$

$$\|\frac{1}{t}\int_0^t T(\tau)x_2 d\tau\| = \|\frac{1}{t}\int_0^t T(\tau)Ax_3 d\tau\| \tag{3.6}$$
$$= \|\frac{1}{t}\int_0^t \frac{d}{dt}[T(\tau)x_3]d\tau\| = \|\frac{1}{t}(T(t)x_3 - x_3)\| \leq \frac{2\|x_3\|}{t}.$$

So that the function $\varphi = T(t)x_3 - x_3$ has all the required properties, and that proved (i). To prove (ii), let X be a Banach space and $B \subseteq X$. We denote by

$$B^{\perp} = \{x^* \in X^*; (x, x^*) = 0, \ \forall x \in B\}. \tag{3.7}$$

Similarly if $B^* \subseteq X^*$, we denote by

$$(B^*)^{\perp} = \{x^{**} \in X^{**}; (x^*, x^{**}) = 0, \ \forall x^* \in B^*\}. \tag{3.8}$$

Now, let $y^* \in \overline{R(A^*)}$. Then there exists $(x_n^*)_{n\in\mathbb{N}}$ in $D(A^*)$ such that $\lim_{n\to\infty} A^*x_n^* = y^*$. Thus, for each $x \in ker(A)$, we have

$$(x, y^*) = \lim_{n\to\infty}(x, A^*x_n^*) = \lim_{n\to\infty}(A^*x, x_n^*) = 0, \tag{3.9}$$

which shows that $y^* \in ker(A)^{\perp}$, so that $\overline{R(A^*)} \subseteq ker(A)^{\perp}$. To prove the converse inclusion, let us assume by contradiction that there exists at least one $y^* \in ker(A)^{\perp}$ such that $y^* \notin \overline{R(A^*)}$. By the Hahn - Banach theoerem, it follows that there exists $x \in X$ (we recall that X is reflexive and identified with X^{**}) such that $(x, y^*) \neq 0$ and $(x, z^*) = 0$ for all $z^* \in \bar{R}(A^*)$. In Particular, we have $(Ax, x^*) = (x, A^*x*) = 0$ for each $x^* \in D(A^*)$. On the other hand, we know that $D(A^*)$ is dense in X^* and therefore $Ax = 0$. So $x \in K(A)$ and consequently $(x, y^*) = 0$ which is impossible. This contradiction can be eliminated only if $\ker(A)^{\perp} \subseteq R(A^*)$ and this complete the proof.

Theorem 3.3

Let X be reflexive and $A : D(A) \subseteq X \to X$ the infinitesimal generator of a C_0-semigroup of contractions, $\{T(t); t \geq 0\}$ such that $A \in \omega\text{-}0RCP_n$. Then $\ker(A)$ and $\overline{R(A)}$ are closed subspaces in X. Moreover, $\ker(A) \cap \overline{R(A)} = \{0\}$, $X = \ker(A) \oplus \overline{R(A)}$ and the projection operator $P : X \to ker(A)$ is well defined, and satisfies $\|P\|_{L(X)} \leq 1$. In addition, each $x \in X$ has a unique decomposition $x = Px + (1 - P)x \in ker(A) \oplus \overline{R(A)}$ and

$\lim_{t\to\infty} \frac{1}{t}\int_0^t T(\tau)x d\tau = Px$.

Proof :

Let us observe that (i) of Theoerem 3.2 implies that $\ker(A) \cap \overline{R(A)} = \{0\}$. So, let $y \in ker(A) \cap \overline{R(A)}$, then there exists $(y_n)_{n\in\mathbb{N}}$ in $R(A)$ such that

$$\lim_{n\to\infty} y_n = y. \tag{3.10}$$

Since $y \in ker(A)$ and $y_n \in R(A)$ for each $n \in \mathbb{N}$ and $A \in \omega\text{-}0RCP_n$, also from (i) of Theorem 3.2, we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t T(\tau)y d\tau = y, \tag{3.11}$$

and

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t T(\tau)y_n d\tau = 0, \tag{3.12}$$

for each $n \in \mathbb{N}$. On the other hand

$$\|\frac{1}{t}\int_0^t T(\tau)y d\tau - \frac{1}{t}\int_0^t T(\tau)y_n d\tau\| \leq \frac{1}{t}\int_0^t \|y - y_n\| d\tau = \|y - y_n\|, \tag{3.13}$$

which implies

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t T(\tau)y d\tau = 0. \tag{3.14}$$

So, if $y = 0$, we need to show that

$$ker(A) \cap \overline{R(A)} = \{0\}. \tag{3.15}$$

Suppose $x \in ker(A) \oplus \overline{R(A)}$, then we have

$$\|Px\| = \lim_{t\to\infty} \|\frac{1}{t}\int_0^t T(\tau)x d\tau\| \leq \|x\|, \tag{3.16}$$

which shows that $\|P\|_{L(X)} \leq 1$. To complete the proof, we have to check that $ker(A) \oplus R(A)$ is dense in X. So, let us assume by contradiction that there exists $x \in X \backslash ker(A) \oplus \overline{R(A)}$ and $x^* \in X^*$, satisfying both $x^* \in (ker(A) + R(A))^{\perp}$ and $(x, x^*) = 1$. Clearly $x^* \in ker(A)^{\perp}$ and $x^* \in R(A)^{\perp}$. By (ii) of Theorem 3.2, e deduced that $x^* \in ker(A^* \cap \overline{R(A^*)}$. On the other hand, since X is reflexive, from Theorem 3.1, we know that A^* is the generator of the C_0-semigroup of contraction, $\{T^*(t); t \geq 0\}$ in X^*. From (3.15), we have

$$ker(A^*) \cap \overline{R(A^*)} = \{0\}, \tag{3.17}$$

and therefore

$$x^* = 0. \tag{3.18}$$

This contradiction can be eliminated only if $ker(A) \oplus \overline{R(A)} = X$ and this complete the proof.

Theorem 3.4

Let X be a Banach space and $(A, D(A))$ the generator of C_0-semigroup of contractions, $\{T(t); t \geq 0\}$ such that $A \in \omega\text{-}0RCP_n$. Then:

(i) Suppose $u : \mathbb{R}_+ \to X$ is a bounded and continuous function and $\lim_{t\to\infty} \frac{1}{t}\int_0^t u(\tau)d\tau = \lim_{\lambda\to 0}\int_0^{\infty} e^{-\lambda s}u(s)ds$, in the sense that, either both limits exist and they are equal, or both do not exist, and the following conditions are equivalent :

(a) $X = kerA \oplus \overline{R(A)}$;

(b) for each $x \in X$ there exists $\lim_{t\to\infty} \frac{1}{t}\int_0^t T(\tau)x d\tau$;

(c) for each $x \in X$ there exists $\lim_{\lambda\to 0} \lambda(\lambda I - A)^{-1}x$; and
(d) for each $x \in D(A)$ there exists $\lim_{\lambda\to 0} A(\lambda I - A)^{-1}x$.
proof :
Assume that there exists

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t u(\tau)d\tau = \ell, \tag{3.19}$$

and let us observe that

$$\lambda\int_0^\infty e^{-\lambda s}u(s)ds = \lambda^2\int_0^\infty e^{-\lambda s}s(\frac{1}{s}\int_0^s u(\tau)d\tau)ds. \tag{3.20}$$

Then

$$|\lambda\int_0^\infty e^{-\lambda s}u(s)ds - \ell| \le \lambda^2\int_0^\infty e^{-\lambda s}s|\frac{1}{s}\int_0^s u(\tau)d\tau - \ell|ds. \tag{3.21}$$

Let $\epsilon > 0$ and fix a $\delta = \delta(\epsilon) > 0$ such that

$$|\frac{1}{s}\int_0^s u(\tau)d\tau - \ell| \le \epsilon \tag{3.22}$$

for each $s \in [\delta, +\infty)$. From (3.22), it follows that

$$\begin{aligned} |\lambda\int_0^\infty e^{-\lambda s}u(s)ds - \ell| &\le \lambda^2\int_0^\delta e^{-\lambda s}s|\frac{1}{s}\int_0^s u(\tau)d\tau - \ell|ds \\ &\quad +\lambda^2\int_\delta^\infty e^{-\lambda s}s|\frac{1}{s}\int_0^s u(\tau)d\tau - \ell|ds \\ &\le \lambda^2\int_0^\delta e^{-\lambda s}s(M+|\ell|)ds + \lambda^2\epsilon\int_\delta^\infty se^{-\lambda s}ds, \end{aligned} \tag{3.23}$$

where $M > 0$ satisfies $\|u(t)\| \le M$ for each $t \in \mathbb{R}_+$. Therefore, a simple computational argument yields

$$|\lambda\int_0^\infty e^{-\lambda s}u(s)ds - \ell| \le (M+|\ell|)(1 - e^{-\lambda\delta} - \lambda\delta e^{-\lambda\delta}) + \epsilon(\lambda\delta e^{-\lambda\delta} + e^{-\lambda\delta}). \tag{3.24}$$

Taking the sup-limit in (3.24) for λ tending to 0 by positive values, we obtained

$$\lim_{\lambda\to 0} sup|\lambda\int_0^\infty e^{-\lambda s}u(s)ds - \ell| \le \epsilon. \tag{3.25}$$

Since $\epsilon > 0$ is arbitrary, this shows that whenever there exists the first limit in (i), then there exists the second one too, and both must be equal. The proof of the fact that existence of the second limit implies the existence of the first one (and by the preceding proof, the fact that both are equal) is much more subtle and follows from a very deep Theorem of Wiener which for the sake of brevity did not find its place here. However, see [13], Theorem 14, and the proof one (i) is complete.

To prove (a) to (d), let us recall the proof of Theorem (3.3), and show first that, for each $x \in ker(A) \oplus R(A)$ and $A \in \omega\text{-}ORCP_n$, there exists

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t T(\tau) d\tau. \tag{3.26}$$

Hence (a) implies (b). From (i) and from the observation that for each $x \in X$, we have

$$\lambda(\lambda I - A)^{-1} x = \lambda \int_0^\infty e^{-\lambda s} T(s) x ds, \tag{3.27}$$

which deduced that (b) implies (c). The fact that (c) implies (d) follows from the identity

$$(\lambda I - A)^{-1} Ax = \lambda(\lambda I - A)^{-1} x - x = A(\lambda I - A)^{-1} x \tag{3.28}$$

for each $x \in D(A)$. Since $D(A)$ is dense in X, from (i) and (3.28), it follows that (d) implies (b). In order to complete the proof, we have merely to show that (b) implies (a). Let us define $P : X \to X$ by

$$Px = \lim_{t\to\infty} \frac{1}{t} \int_0^t T(\tau) x d\tau \tag{3.29}$$

for each $x \in X$. We have $X = R(P) + R(I - P)$ and $PT(t)x = T(t)Px = Px$, for each $x \in X$ and each $t \geq 0$. In addition, let us remark that $P^2 = P$, $R(P) = ker(A)$, $R(I - P) = ker(P)$, and $\overline{R(A)} \subseteq ker(P)$ for all $A \in \omega\text{-}ORCP_n$. Then, by (3.28) and (c) (which is implied by (b)), it follows that

$$\lim_{\lambda\to 0} A(\lambda I - A)^{-1} x = Px - x = -x \tag{3.30}$$

and therefore $x \in \overline{R(A)}$. Hence the proof is complete.

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