ASIAN JOURNAL OF MATHEMATICS AND APPLICATIONS
Volume 2020, Article ID ama0559, 14 pages
ISSN 2307-7743
http://scienceasia.asia

GROWTH OF A CLASS OF ITERATION OF K-ENTIRE FUNCTIONS

DIBYENDU BANERJEE[1,*], SUMANTA GHOSH[2]

[1]*Department of Mathematics, Visva-Bharati, Santiniketan-731235, West Bengal, India*
[2]*Ranaghat P.C.High School, Ranaghat-741201, Nadia, West Bengal, India*

[*]Correspondence: dibyendu192@rediffmail.com

Abstract. In this paper we generalise a result of Sun [7] to n-th iterations of k-entire functions.

1. INTRODUCTION AND NOTATION

For two entire functions $f(z)$ and $g(z)$ Lahiri and Banerjee [5] form the following iterations.

Let

$$\begin{aligned}
f_1(z) &= f(z)\\
f_2(z) &= f(g(z)) = f(g_1(z))\\
f_3(z) &= f(g(f(z))) = f(g(f_1(z)))\\
&\vdots\\
f_n(z) &= f(g(f(g...(f(z) \text{ or } g(z) \text{ according as } n \text{ is odd or even})...)))\\
&= f(g_{n-1}(z)) = f(g(f_{n-2}(z)))
\end{aligned}$$

and so are

$$\begin{aligned}
g_1(z) &= g(z)\\
g_2(z) &= g(f(z)) = g(f_1(z))\\
g_3(z) &= g(f_2(z)) = g(f(g_1(z)))\\
&\vdots\\
g_n(z) &= g(f_{n-1}(z)) = g(f(g_{n-2}(z))).
\end{aligned}$$

Keywords: entire function, growth, iteration.

Clearly all $f_n(z)$ and $g_n(z)$ are entire functions.

Notation 1.1. *Let $f(z)$ and $g(z)$ be entire functions. Banerjee and Dutta [1] used the notations $M_{f_1}(r)$, $M_{f_2}(r)$, $M_{f_3}(r)$ etc., to mean $M(r,f)$, $M(M(r,f),g)$, $M(M(M(r,f),g),f)$ etc., respectively and $F(r)=O^*(G(r))$ to mean that there exist two positive constants K_1 and K_2 such that $K_1 \leq \frac{F(r)}{G(r)} \leq K_2$ for any r big enough.*

In 2003, Sun [7] proved the following theorem.

Theorem 1.2. *Let f_1, f_2, and g_1, g_2 be four transcendental entire functions with $T(r,f_1)=O^*\left((\log r)^{\nu} e^{(\log r)^{\alpha}}\right)$ and $T(r,g_1)=O^*\left((\log r)^{\beta}\right)$.*

If $T(r,f_1)\sim T(r,f_2)$ and $T(r,g_1)\sim T(r,g_2)$ $(r\to\infty)$, then

$$T(r,f_1(g_1))\sim T(r,f_2(g_2)) \quad (r\to\infty,\ r\notin E),$$

where $\nu>0$, $0<\alpha<1$, $\beta>1$ and $\alpha\beta<1$ and E is a set of finite logarithmic measure.

In 2011, Banerjee and Dutta [1] extend Theorem 1.2 for iterated entire functions in the following manner.

Theorem 1.3. *Let f, g and u, v be four transcendental entire functions with $T(r,f)\sim T(r,u)$, $T(r,g)\sim T(r,v)$, $T(r,f)=O^*\left((\log r)^{\nu} e^{(\log r)^{\alpha}}\right)$ $(0<\alpha<1,\nu>0)$ and $T(r,g)=O^*\left((\log r)^{\beta}\right)$ where $\beta>1$ is a constant, then $T(r,f_n)\sim T(r,u_n)$ for $n\geq 2$, where $u_n(z)=u(v(u(v...(u(z)$ or $v(z))...)))$ according as n is odd or even.*

Theorem 1.4. *Let f, g and u, v be four transcendental entire functions with $T(r,f)\sim T(r,u)$, $T(r,g)\sim T(r,v)$, $T(r,f)=O^*\left((\log r)^{\beta}\right)$ and $T(r,g)=O^*\left((\log r)^{\beta}\right)$ where $\beta>1$ is a constant, then $T(r,f_n)\sim T(r,u_n)$.*

In [6], Niino and Suita proved the following theorem.

Theorem 1.5. *Let $f(z)$ and $g(z)$ be entire functions. If $M(r,g)>\frac{2+\varepsilon}{\varepsilon}|g(0)|$ for any $\varepsilon>0$, then we have*

$$T(r,f(g))\leq(1+\varepsilon)T(M(r,g),f).$$

In particular if $g(0)=0$, then

$$T(r,f(g))\leq T(M(r,g),f) \text{ for all } r>0.$$

As a generalisation of Theorem 1.5, Banerjee and Dutta [1] proved the following.

Theorem 1.6. *Let $f(z)$, $g(z)$ be two entire functions. Then we have*

$$T(R_2,f)\leq T(r,f_n)\leq T(R_3,f)$$

where $|f(z)| > R_1 > \frac{2+\varepsilon}{\varepsilon} |f(0)|$ *and* $|g(z)| > R_2 > \frac{2+\varepsilon}{\varepsilon} |g(0)|$, $R_3 = \max\{M_{f_{n-1}}(r), M_{g_{n-1}}(r)\}$ *for sufficiently large values of* r *and any integer* $n \geq 2$.

After this in [2], Banerjee and Mandal considered three entire functions $f(z)$, $g(z)$ and $h(z)$ and formed the iterations of $f(z)$ with respect to $g(z)$ and $h(z)$ as follows.

$$\begin{aligned}
f_1(z) &= f(z) \\
f_2(z) &= f(g(z)) = f(g_1(z)) \\
f_3(z) &= f(g(h(z))) = f(g(h_1(z))) = f(g_2(z)) \\
f_4(z) &= f(g(h(f(z)))) = f(g(h_2(z))) = f(g_3(z)) \\
&\vdots \\
f_n(z) &= f(g(h(f..(f(z) \text{ or } g(z) \text{ or } h(z) \text{ according as } n = 3m-2 \text{ or } 3m-1 \\
&\quad \text{or } 3m)...))) \\
&= f(g_{n-1}(z)) = f(g(h_{n-2}(z))).
\end{aligned}$$

And

$$\begin{aligned}
g_1(z) &= g(z) \\
g_2(z) &= g(h(z)) = g(h_1(z)) \\
g_3(z) &= g(h(f(z))) = g(h(f_1(z))) = g(h_2(z)) \\
g_4(z) &= g(h(f(g(z)))) = g(h(f_2(z))) = g(h_3(z)) \\
&\vdots \\
g_n(z) &= g(h(f(g...(g(z) \text{ or } h(z) \text{ or } f(z) \text{ according as } n = 3m-2 \text{ or } 3m-1 \\
&\quad \text{or } 3m)...))) \\
&= g(h_{n-1}(z)) = g(h(f_{n-2}(z))).
\end{aligned}$$

Similarly,

$$\begin{aligned}
h_1(z) &= h(z) \\
h_2(z) &= h(f(z)) = h(f_1(z)) \\
h_3(z) &= h(f(g(z))) = h(f(g_1(z))) = h(f_2(z)) \\
h_4(z) &= h(f(g(h(z)))) = h(f(g_2(z))) = h(f_3(z)) \\
&\vdots \\
h_n(z) &= h(f(g(h...(h(z) \text{ or } f(z) \text{ or } g(z) \text{ according as } n = 3m-2 \text{ or } 3m-1 \\
&\quad \text{or } 3m)...))) \\
&= h(f_{n-1}(z)) = h(f(g_{n-2}(z))).
\end{aligned}$$

Clearly f_n, g_n and h_n are all entire functions.

Notation 1.7. *For three entire functions $f(z)$, $g(z)$ and $h(z)$ Banerjee and Ghosh* [3] *used the notations $M_{f_1}(r)$, $M_{f_2}(r)$, $M_{f_3}(r)$, $M_{f_4}(r)$ etc., to mean $M(r,f)$, $M(M(r,f),h)$, $M(M(M(r,f),h),g)$, $M(M(M(M(r,f),h),g),f)$ etc., respectively. Similarly $M_{g_1}(r)$, $M_{g_2}(r)$, $M_{g_3}(r)$, $M_{g_4}(r)$ etc., to mean $M(r,g)$, $M(M(r,g),f)$, $M(M(M(r,g),f),h)$, $M(M(M(M(r,g),f),h),g)$ etc., respectively and $M_{h_1}(r)$, $M_{h_2}(r)$, $M_{h_3}(r)$, $M_{h_4}(r)$ etc., to mean $M(r,h)$, $M(M(r,h),g)$, $M(M(M(r,h),g),f)$, $M(M(M(M(r,h),g),f),h)$ etc., respectively. Also $F(r)=O^*(G(r))$ to mean that there exist two positive constants K_1 and K_2 such that $K_1 \le \frac{F(r)}{G(r)} \le K_2$ for any r big enough.*

Considering such type of iterations in a recent paper [3], Banerjee and Ghosh extended the result of Banerjee and Dutta [1] in this direction.

Theorem 1.8. *Let $f(z)$, $g(z)$ and $h(z)$ be three entire functions. Then we have*

$$T(R_2,f) \le T(r,f_n) \le T(R_4,f)$$

where $|f(z)| > R_1 > \frac{2+\varepsilon}{\varepsilon}|f(0)|$, $|g(z)| > R_2 > \frac{2+\varepsilon}{\varepsilon}|g(0)|$, $|h(z)| > R_3 > \frac{2+\varepsilon}{\varepsilon}|h(0)|$ and $R_4 = \max\{M_{f_{n-1}}(r), M_{g_{n-1}}(r), M_{h_{n-1}}(r)\}$ for sufficiently large values of r and any integer $n \ge 3$.

Theorem 1.9. *Let f, g, h; u, v, w be six transcendental entire functions with $T(r,f) \sim T(r,u)$, $T(r,g) \sim T(r,v)$, $T(r,h) \sim T(r,w)$, $T(r,f) = O^*\left((\log r)^{\nu} e^{(\log r)^{\alpha}}\right)$ $(0<\alpha<1, \nu>0)$, $T(r,g) = O^*\left((\log r)^{\beta}\right)$ where $\beta > 1$ is a constant and $T(r,h) = O^*\left((\log r)^{\lambda}\right)$ where $\lambda > 1$ is a constant. Then $T(r,f_n) \sim T(r,u_n)$ for $n \ge 3$, where $u_n(z) = u(v(w(u\ldots(u(z)$ or $v(z)$ or $w(z)\ldots))))$ according as $n = 3m-2$ or $3m-1$ or $3m$, $m \in \mathbb{N}$.*

Theorem 1.10. *Let f, g, h; u, v, w be six transcendental entire functions with $T(r,f) \sim T(r,u)$, $T(r,g) \sim T(r,v)$, $T(r,h) \sim T(r,w)$, $T(r,f) = O^*\left((\log r)^{\beta}\right)$, $T(r,g) = O^*\left((\log r)^{\beta}\right)$ and $T(r,h) = O^*\left((\log r)^{\beta}\right)$ where $\beta > 1$ is a constant. Then $T(r,f_n) \sim T(r,u_n)$ for $n \ge 3$,*

where $u_n(z) = u(v(w(u\ldots(u(z)$ or $v(z)$ or $w(z)\ldots))))$ according as $n = 3m-2$ or $3m-1$ or $3m$, $m \in \mathbb{N}$.

At the present paper we consider k entire functions $f_1(z)$, $f_2(z)$, $f_3(z)$, ..., $f_k(z)$ and following the iteration process of Banerjee and Ghosh [4] (defined below) generalise the result of Sun [7].

Let

$$\begin{aligned}
F_1^1(z) &= f_1(z) \\
F_2^1(z) &= f_1(f_2(z)) = f_1\left(F_1^2(z)\right) \\
F_3^1(z) &= f_1(f_2(f_3(z))) = f_1\left(f_2\left(F_1^3(z)\right)\right) = f_1(F_2^2(z)) \\
&\vdots \\
F_k^1(z) &= f_1(f_2(...f_k(z))) = f_1\left(F_{k-1}^2(z)\right) = f_1\left(f_2\left(F_{k-2}^3(z)\right)\right) \\
&= ... = f_1\left(f_2\left(...f_{k-1}(F_1^k(z))\right)\right) \\
&\vdots \\
F_n^1(z) &= f_1(f_2(f_3...(f_1(z) \text{ or } f_2(z) \text{ or...or } f_k(z) \text{ according} \\
\text{as } n &= km-(k-1) \text{ or } km-(k-2) \text{ or ...or } km)...)) \\
&= f_1\left(F_{n-1}^2(z)\right) \\
&= f_1\left(f_2\left(F_{n-2}^3(z)\right)\right) \\
&= ...... \\
&= f_1\left(f_2\left(...\left(f_{k-1}\left(F_{n-(k-1)}^k(z)\right)\right)\right)\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
F_1^2(z) &= f_2(z) \\
F_2^2(z) &= f_2(f_3(z)) = f_2\left(F_1^3(z)\right) \\
F_3^2(z) &= f_2(f_3(f_4(z))) = f_2\left(f_3\left(F_1^4(z)\right)\right) = f_2(F_2^2(z)) \\
&\vdots \\
F_k^2(z) &= f_2(f_3(...f_k(f_1(z)))) = f_2\left(F_{k-1}^3(z)\right) = ... \\
&= f_2\left(f_3\left(...f_k\left(F_1^1(z)\right)\right)\right) \\
&\vdots \\
F_n^2(z) &= f_2(f_3(f_4...(f_2(z) \text{ or } f_3(z) \text{ or ...or } f_k(z) \text{ or } f_1(z) \text{ according} \\
\text{as } n &= km-(k-1) \text{ or } km-(k-2) \text{ or ...or } (km-1) \text{ or } km)...)) \\
&= f_2\left(F_{n-1}^3(z)\right) \\
&= f_2\left(f_3\left(F_{n-2}^4(z)\right)\right) \\
&= ...... \\
&= f_2\left(f_3\left(...\left(f_k\left(F_{n-(k-1)}^1(z)\right)\right)\right)\right).
\end{aligned}$$

$$\vdots$$

And

$$\begin{aligned}
F_1^k(z) &= f_k(z) \\
F_2^k(z) &= f_k(f_1(z)) = f_k\left(F_1^1(z)\right) \\
F_3^k(z) &= f_k(f_1(f_2(z))) = f_k\left(f_1\left(F_1^2(z)\right)\right) = f_k(F_2^1(z)) \\
&\vdots \\
F_k^k(z) &= f_k(f_1(...f_{k-2}(f_{k-1}(z)))) = ... \\
&= f_k\left(f_1\left(...f_{k-2}\left(F_1^{k-1}(z)\right)\right)\right) \\
&\vdots \\
F_n^k(z) &= f_k(f_1(f_2...(f_k(z) \text{ or } f_1(z) \text{ or } ...\text{or } f_{k-1}(z) \text{ according} \\
\text{as } n &= km-(k-1) \text{ or } km-(k-2) \text{ or } ...\text{or } km)...)) \\
&= f_k\left(F_{n-1}^1(z)\right) \\
&= f_k\left(f_1\left(F_{n-2}^2(z)\right)\right) \\
&= .... \\
&= f_k\left(f_1\left(...\left(f_{k-2}\left(F_{n-(k-1)}^{k-1}(z)\right)\right)\right)\right).
\end{aligned}$$

Clearly all F_n^1, $F_n^2, ...,$ F_n^k are entire functions.

Notation 1.11. *Let $f_1(z)$, $f_2(z), ...,$ $f_k(z)$ be entire functions. We use the notations $M_{F_1^1}(r)$, $M_{F_2^1}(r)$, $M_{F_3^1}(r)$,..., $M_{F_k^1}(r)$ etc., to mean $M(r, f_1)$, $M(M(r, f_1), f_k)$, $M(M(M(r, f_1), f_k), f_{k-1})$, ...,$M(...(M(M(M(r, f_1), f_k), f_{k-1}), ...), f_2)$ etc., respectively. Similarly we use the notations $M_{F_1^2}(r)$, $M_{F_2^2}(r)$, $M_{F_3^2}(r)$,..., $M_{F_k^2}(r)$ etc., to mean $M(r, f_2)$, $M(M(r, f_2), f_1)$, $M(M(M(r, f_2), f_1), f_k)$, ..., $M(...(M(M(M(r, f_2), f_1), f_k), ...), f_3)$ etc., respectively and proceding in this way we use the notations $M_{F_1^k}(r)$, $M_{F_2^k}(r)$, $M_{F_3^k}(r)$,..., $M_{F_4^k}(r)$ etc., to mean $M(r, f_k)$, $M(M(r, f_k), f_{k-1})$, $M(M(M(r, f_k), f_{k-1}), f_{k-2})$,...,*

$M(...(M(M(M(r, f_k), f_{k-1}), f_{k-2}), ...), f_1)$ etc., respectively and $F(r) = O^*(G(r))$ to mean that there exist two positive constants K_1 and K_2 such that $K_1 \leq \frac{F(r)}{G(r)} \leq K_2$ for any r big enough.

2. LEMMAS

Lemma 2.1. [6] *Let $g(z)$ and $f(z)$ be two entire functions. Suppose that $|g(z)| > R > |g(0)|$ on the circumference $\{|z| = r\}$ for some $r > 0$. Then we have*

$$T(r, f(g)) \geq \frac{R - |g(0)|}{R + |g(0)|} T(R, f).$$

Lemma 2.2. [7] *Let f be a transcendental entire function with*

$$T(r, f) = O^*\left((\log r)^{\beta} e^{(\log r)^{\alpha}}\right) \qquad (0 < \alpha < 1, \beta > 0).$$

Then

$$\begin{aligned} T(r, f) &\sim \log M(r, f) \qquad (r \to \infty, r \notin E) \quad \text{and} \\ T(\sigma r, f) &\sim T(r, f) \qquad (r \to \infty, \sigma \geq 2, r \notin E), \end{aligned}$$

where E is a set of finite logarithmic measure.

Lemma 2.3. [1] *Let f be a transcendental entire function with $T(r, f) = O^*\left((\log r)^{\beta}\right)$ where $\beta > 1$. Then*

$$\begin{aligned} T(r, f) &\sim \log M(r, f) \qquad (r \to \infty, r \notin E) \quad \text{and} \\ T(\sigma r, f) &\sim T(r, f) \qquad (r \to \infty, \sigma \geq 2, r \notin E), \end{aligned}$$

where E is a set of finite logarithmic measure.

Lemma 2.4. [1] *Let f_1 and f_2 be two entire functions with $T(r, f_1) = O^*\left((\log r)^{\nu} e^{(\log r)^{\alpha}}\right)$ where $\nu > 1$, $0 < \alpha < 1$ and $T(r, f_1) \sim T(r, f_2)$ then $M(r, f_1) \sim M(r, f_2)$.*

Lemma 2.5. [1] *Let f_1 and f_2 be two entire functions with $T(r, f_1) = O^*\left((\log r)^{\beta}\right)$ where $\beta > 1$ and $T(r, f_1) \sim T(r, f_2)$ then $M(r, f_1) \sim M(r, f_2)$.*

3. MAIN RESULTS

Theorem 3.1. *Let $f_1, f_2, ..., f_k$ be k entire functions. Then we have*

$$T(R_2, f_1) \leq T\left(r, F_n^1\right) \leq T(R_{k+1}, f_1) \tag{3.1}$$

where $|f_i(z)| > R_i > \frac{2+\varepsilon}{\varepsilon} |f_i(0)|$,$i = 1, 2, ..., k$ and
$R_{k+1} = \max\left\{M_{F_{n-1}^1}(r),\ M_{F_{n-1}^2}(r), ...,\ M_{F_{n-1}^{k-1}}(r),\ M_{F_{n-1}^k}(r)\right\}$ for sufficiently large values of r and any integer $n \geq k$.

Proof. We first prove the 2nd inequality. By Theorem 1.5 and for $\varepsilon > 0$ arbitrary small we have when $n = km$, $m \in \mathbb{N}$

$$\begin{aligned}
T\left(r, F_n^1\right) &= T\left(r, F_{n-1}^1\left(f_k\right)\right) \\
&\leq (1+\varepsilon)\, T\left(M\left(r, f_k\right), F_{n-1}^1\right) \\
&= (1+\varepsilon)\, T\left(M_{F_1^k}\left(r\right), F_{n-2}^1\left(f_{k-1}\right)\right) \\
&\leq (1+\varepsilon)^2\, T\left(M_{F_2^k}\left(r\right), F_{n-3}^1\left(f_{k-2}\right)\right) \\
&\leq (1+\varepsilon)^3\, T\left(M_{F_3^k}\left(r\right), F_{n-4}^1\left(f_{k-3}\right)\right) \\
&= (1+\varepsilon)^4\, T\left(M_{F_4^k}\left(r\right), F_{n-4}^1\right) \\
&\vdots \\
&\leq (1+\varepsilon)^{n-1}\, T\left(M_{F_{n-1}^k}\left(r\right), f_1\right) \\
&\leq (1+\varepsilon)^{n-1}\, T\left(R_{k+1}, f_1\right).
\end{aligned}$$

When $n = km - 1$, $m \in \mathbb{N}$

$$\begin{aligned}
T\left(r, F_n^1\right) &= T\left(r, F_{n-1}^1\left(f_{k-1}\right)\right) \\
&\leq (1+\varepsilon)\, T\left(M\left(r, f_{k-1}\right), F_{n-1}^1\right) \\
&= (1+\varepsilon)\, T\left(M_{F_1^{k-1}}\left(r\right), F_{n-2}^1\left(f_{k-2}\right)\right) \\
&\leq (1+\varepsilon)^2\, T\left(M_{F_2^{k-1}}\left(r\right), F_{n-3}^1\left(f_{k-3}\right)\right) \\
&\leq (1+\varepsilon)^3\, T\left(M_{F_3^{k-1}}\left(r\right), F_{n-4}^1\left(f_{k-4}\right)\right) \\
&= (1+\varepsilon)^4\, T\left(M_{F_4^{k-1}}\left(r\right), F_{n-4}^1\right) \\
&\vdots \\
&\leq (1+\varepsilon)^{n-1}\, T\left(M_{F_{n-1}^{k-1}}\left(r\right), f_1\right) \\
&\leq (1+\varepsilon)^{n-1}\, T\left(R_{k+1}, f_1\right).
\end{aligned}$$

.

.

.

Similarly for $n = km - (k-1)$

$$\begin{aligned}
T\left(r, F_n^1\right) &= T\left(r, F_{n-1}^1\left(f_1\right)\right) \\
&\leq (1+\varepsilon) T\left(M\left(r, f_1\right), F_{n-1}^1\right) \\
&= (1+\varepsilon) T\left(M_{F_1^1}(r), F_{n-2}^1\left(f_k\right)\right) \\
&\leq (1+\varepsilon)^2 T\left(M_{F_2^1}(r), F_{n-3}^1\left(f_{k-1}\right)\right) \\
&\leq (1+\varepsilon)^3 T\left(M_{F_3^1}(r), F_{n-4}^1\left(f_{k-2}\right)\right) \\
&= (1+\varepsilon)^4 T\left(M_{F_4^1}(r), F_{n-4}^1\right) \\
&\vdots \\
&\leq (1+\varepsilon)^{n-1} T\left(M_{F_{n-1}^1}(r), f_1\right) \\
&\leq (1+\varepsilon)^{n-1} T\left(R_{k+1}, f_1\right).
\end{aligned}$$

Therefore $T\left(r, F_n^1\right) \leq (1+\varepsilon)^{n-1} T\left(R_{k+1}, f_1\right)$ for any integer $n \geq k$.
Since $\varepsilon > 0$ is arbitrary, for sufficiently large values of r we have

$$T\left(r, F_n^1\right) \leq T\left(R_{k+1}, f_1\right). \tag{3.2}$$

We now prove the 1st inequality by using Lemma 2.1.
When $n = km$, $m \in \mathbb{N}$

$$\begin{aligned}
T\left(r, F_n^1\right) &= T\left(r, F_{n-1}^1\left(f_k\right)\right) \\
&\geq \frac{R_k - \left|f_k(0)\right|}{R_k + \left|f_k(0)\right|} T\left(R_k, F_{n-1}^1\right) \\
&> (1-\varepsilon) T\left(R_k, F_{n-2}^1\left(f_{k-1}\right)\right), \text{since } \ R_k > \frac{2+\varepsilon}{\varepsilon}\left|f_k(0)\right| > \frac{2-\varepsilon}{\varepsilon}\left|f_k(0)\right| \\
&\geq (1-\varepsilon) \frac{R_{k-1} - \left|f_{k-1}(0)\right|}{R_{k-1} + \left|f_{k-1}(0)\right|} T\left(R_{k-1}, F_{n-2}^1\right) \\
&> (1-\varepsilon)^2 T\left(R_{k-1}, F_{n-2}^1\right) \\
&\geq (1-\varepsilon)^3 T\left(R_{k-2}, F_{n-3}^1\right) \\
&\vdots \\
&\geq (1-\varepsilon)^{n-2} T\left(R_3, f_1\left(f_2\right)\right) \\
&\geq (1-\varepsilon)^{n-1} T\left(R_2, f_1\right).
\end{aligned}$$

When $n = km - 1$

$$\begin{aligned}
T\left(r, F_n^1\right) &= T\left(r, F_{n-1}^1\left(f_{k-1}\right)\right) \\
&\geq \frac{R_{k-1} - \left|f_{k-1}(0)\right|}{R_{k-1} + \left|f_{k-1}(0)\right|} T\left(R_{k-1}, F_{n-1}^1\right) \\
&> (1-\varepsilon) T\left(R_{k-1}, F_{n-2}^1\left(f_{k-2}\right)\right) \\
&\geq (1-\varepsilon) \frac{R_{k-2} - \left|f_{k-2}(0)\right|}{R_{k-2} + \left|f_{k-2}(0)\right|} T\left(R_{k-2}, F_{n-2}^1\right) \\
&> (1-\varepsilon)^2 T\left(R_{k-2}, F_{n-2}^1\right) \\
&\geq (1-\varepsilon)^3 T\left(R_{k-3}, F_{n-3}^1\right) \\
&\ \ \vdots \\
&\geq (1-\varepsilon)^{n-2} T\left(R_3, f_1\left(f_2\right)\right) \\
&\geq (1-\varepsilon)^{n-1} T\left(R_2, f_1\right).
\end{aligned}$$

.

.

.

Similarly for $n = km - (k-1)$

$$\begin{aligned}
T\left(r, F_n^1\right) &= T\left(r, F_{n-1}^1\left(f_1\right)\right) \\
&\geq \frac{R_1 - \left|f_1(0)\right|}{R_1 + \left|f_1(0)\right|} T\left(R_1, F_{n-1}^1\right) \\
&> (1-\varepsilon) T\left(R_1, F_{n-2}^1\left(f_k\right)\right) \\
&\geq (1-\varepsilon) \frac{R_k - \left|f_k(0)\right|}{R_k + \left|f_k(0)\right|} T\left(R_k, F_{n-2}^1\right) \\
&> (1-\varepsilon)^2 T\left(R_k, F_{n-2}^1\right) \\
&\geq (1-\varepsilon)^3 T\left(R_{k-1}, F_{n-3}^1\right) \\
&\ \ \vdots \\
&\geq (1-\varepsilon)^{n-2} T\left(R_3, f_1\left(f_2\right)\right) \\
&\geq (1-\varepsilon)^{n-1} T\left(R_2, f_1\right).
\end{aligned}$$

So

$$T\left(r, F_n^1\right) \geq (1-\varepsilon)^{n-1} T\left(R_2, f_1\right).$$

Since $\varepsilon > 0$ is arbitrary, we have for sufficiently large values of r

$$T\left(r, F_n^1\right) \geq T\left(R_2, f_1\right). \tag{3.3}$$

Hence from (3.2) and (3.3), we have (3.1) .

This proves the theorem. □

Theorem 3.2. *Let $f_1, f_2, ..., f_k$ and $u_1, u_2, ..., u_k$ be $2k$ transcendental entire functions with $T(r, f_i) \sim T(r, u_i)$, $i = 1, 2, ..., k$, $T(r, f_1) = O^*\left((\log r)^{\nu} e^{(\log r)^{\beta_1}}\right)$ $(0 < \beta_1 < 1, \nu > 1)$ and $T(r, f_i) = O^*\left((\log r)^{\beta_i}\right)$, $i = 2, 3, ..., k$, where each $\beta_i > 1$ is a constant. Also let $R'_l \geq 2R_l, l = 1, 2, ..., k$ for $|f_i(z)| > R_i > \frac{2+\varepsilon}{\varepsilon}|f_i(0)|$, $i = 1, 2, ..., k$; $|u_j(z)| > R'_j > \frac{2+\varepsilon}{\varepsilon}|u_j(0)|, j = 1, 2, ..., k$. Then $T(r, F_n^1) \sim T(r, U_n^1)$ for $n \geq k$ where $U_n^1(z) = u_1(u_2(u_3(...(u_1(z)$ or $u_2(z)$ or...or $u_k(z)...))))$ according as $n = km - 1$ or $km - (k-2)$ or ...or km, $m \in \mathbb{N}$.*

Proof. We have from Theorem 3.1

$$T(R_2, f_1) \leq T\left(r, F_n^1\right) \leq T(R_{k+1}, f_1) \tag{3.4}$$

$$\text{and} \quad T(R'_2, u_1) \leq T\left(r, U_n^1\right) \leq T\left(R'_{k+1}, u_1\right) \tag{3.5}$$

where $R_{k+1} = \max\left\{M_{F_{n-1}^1}(r), M_{F_{n-1}^2}(r), ..., M_{F_{n-1}^k}(r)\right\}$ and
$R'_{k+1} = \max\left\{M_{U_{n-1}^1}(r), M_{U_{n-1}^2}(r), ..., M_{U_{n-1}^k}(r)\right\}$ for sufficiently large values of r. Also by Lemma 2.2,

$$T(R_2, f_1) \sim T(R'_2, f_1).$$

Since

$$T(R'_2, f_1) \sim T(R'_2, u_1) \qquad \text{so}$$

$$T(R_2, f_1) \sim T(R'_2, u_1) \quad (R_2 \to \infty,\ R_2 \notin E) \tag{3.6}$$

where E is a set of finite logarithmic measure.

Using Lemma 2.4, we have

$$M(r, f_1) \sim M(r, u_1).$$

Also using Lemma 2.5 we have

$$M(M(r, f_1), f_k) \sim M(M(r, u_1), u_k) \quad (r \to \infty).$$

Similarly $M(M(M(r, f_1), f_k), f_{k-1}) \sim M(M(M(r, u_1), u_k), u_{k-1}) \quad (r \to \infty)$.

So for $n = km - (k-1)$, $m \in \mathbb{N}$ we have

$$M_{F_{n-1}^1}(r) \sim M_{U_{n-1}^1}(r) \quad (r \to \infty). \tag{3.7}$$

Similarly for $n = km - (k-2)$, $m \in \mathbb{N}$ we have

$$M_{F_{n-1}^2}(r) \sim M_{U_{n-1}^2}(r) \quad (r \to \infty). \tag{3.8}$$

$$\vdots$$

And for $n = km$, $m \in \mathbb{N}$ we have

$$M_{F_{n-1}^k}(r) \sim M_{U_{n-1}^k}(r) \quad (r \to \infty). \tag{3.9}$$

From (3.7), (3.8) and (3.9) and for $n \geq k$, $n \in \mathbb{N}$ we obtain $R_{k+1} \sim R'_{k+1}$ for large r.

Since $T(r, f_1) \sim T(r, u_1)$ so we have

$$T(R_{k+1}, f_1) \sim T\left(R'_{k+1}, u_1\right) \quad . \tag{3.10}$$

So from (3.4), (3.5), (3.6) and (3.10) we get

$$T\left(r, F_n^1\right) \sim T\left(r, U_n^1\right).$$

This completes the proof. □

Theorem 3.3. *Let $f_1, f_2, ..., f_k$ and $u_1, u_2, ..., u_k$ be $2k$ transcendental entire functions with $T(r, f_i) \sim T(r, u_i)$, $i = 1, 2, ..., k$, $T(r, f_i) = O^*\left((\log r)^{\beta_i}\right)$, $i = 1, 2, 3, ..., k$, where each $\beta_i > 1$ is a constant. Also let $R'_l \geq 2R_l, l = 1, 2, ..., k$ for $|f_i(z)| > R_i > \frac{2+\varepsilon}{\varepsilon}|f_i(0)|$, $i = 1, 2, ..., k$; $|u_j(z)| > R'_j > \frac{2+\varepsilon}{\varepsilon}|u_j(0)|$, $j = 1, 2, ..., k$. Then $T(r, F_n^1) \sim T(r, U_n^1)$ for $n \geq k$ where $U_n^1(z) = u_1(u_2(u_3(...(u_1(z)$ or $u_2(z)$ or...or $u_k(z)...))))$ according as $n = km - 1$ or $km - (k-2)$ or ...or km, $m \in \mathbb{N}$.*

Proof. Under the given hypothesis here also we obtain the same inequalities (3.4) and (3.5) of Theorem 3.2

Also by Lemma 2.3,

$$T(R_2, f_1) \sim T(R'_2, f_1).$$

Since

$$T(R'_2, f_1) \sim T(R'_2, u_1) \quad \text{so}$$

$$T(R_2, f_1) \sim T(R'_2, u_1) \qquad (R_2 \to \infty,\ R_2 \notin E) \tag{3.11}$$

where E is a set of finite logarithmic measure.

Now using Lemma 2.5, we have

$$M(r, f_1) \sim M(r, u_1).$$

So

$$M(M(r, f_1), f_k) \sim M(M(r, u_1), u_k) \qquad (r \to \infty).$$

Similarly $M(M(M(r, f_1), f_k), f_{k-1}) \sim M(M(M(r, u_1), u_k), u_{k-1}) \quad (r \to \infty)$.

So for $n = km - (k-1)$, $m \in \mathbb{N}$ we have

$$M_{F_{n-1}^1}(r) \sim M_{U_{n-1}^1}(r) \qquad (r \to \infty). \tag{3.12}$$

Similarly for $n = km - (k-2)$, $m \in \mathbb{N}$ we have

$$M_{F_{n-1}^2}(r) \sim M_{U_{n-1}^2}(r) \qquad (r \to \infty). \tag{3.13}$$

$$\vdots$$

And for $n = km$, $m \in \mathbb{N}$ we have

$$M_{F_{n-1}^k}(r) \sim M_{U_{n-1}^k}(r) \qquad (r \to \infty). \tag{3.14}$$

From (3.12), (3.13) and (3.14) and for $n \geq k$, $n \in \mathbb{N}$ we obtain $R_{k+1} \sim R'_{k+1}$ for large r. Since $T(r, f_1) \sim T(r, u_1)$ so we have

$$T(R_{k+1}, f_1) \sim T\left(R'_{k+1}, u_1\right) \quad . \tag{3.15}$$

So from (3.4), (3.5), (3.11) and (3.15) we get

$$T\left(r, F_n^1\right) \sim T\left(r, U_n^1\right).$$

This completes the proof. □

Note 3.4. *The conditions of Theorem 3.3 are not strictly sharp. Which are illustrated by the following example.*

Example 3.5. *Let* $f_1(z) = e^z$, $f_2(z) = 2z$, $f_3(z) = 3z$, $f_4(z) = 4z, ..., f_k(z) = kz$ *and* $u_1(z) = 2e^z$, $u_2(z) = 4z$, $u_3(z) = 6z$, $u_4(z) = 8z, ..., u_k(z) = 2kz$. *Here*

$$\begin{aligned} F_k^1(z) &= f_1(f_2(f_3(...(f_k(z))))) \\ &= f_1(f_2(f_3(...(kz)))) \\ &= f_1(k!z) \\ &= e^{k!z} \end{aligned}$$

and

$$\begin{aligned} U_k^1(z) &= u_1(u_2(u_3(...(u_k(z))))) \\ &= u_1(u_2(u_3(...(2kz)))) \\ &= u_1\left(2^{k-1}.k!z\right) \\ &= 2e^{2^{k-1}.k!z}. \end{aligned}$$

Also

$$\begin{aligned} T(r, f_1) &= \frac{r}{\pi}, \quad T(r, u_1) = \frac{r}{\pi} + \log 2 \\ T(r, f_2) &= \log r + \log 2, \quad T(r, u_2) = \log r + \log 4 \\ T(r, f_3) &= \log r + \log 3, \quad T(r, u_3) = \log r + \log 6 \\ &\vdots \\ T(r, f_k) &= \log r + \log k, \quad T(r, u_k) = \log r + \log 2k. \end{aligned}$$

Therefore

$$T(r, f_k) \sim T(r, u_k) \text{ for each } k \text{ as } r \to \infty.$$

$$\text{But since } T\left(r, F_k^1\right) = \frac{r}{k!\pi} \text{ and } T\left(r, U_k^1\right) = \frac{r}{2^{k-1}.k!\pi} + \log 2,$$

so

$$T\left(r, F_k^1\right) \nsim T\left(r, U_k^1\right) \quad \text{as } r \to \infty.$$

Therefore $T\left(r, F_n^1\right) \sim T\left(r, U_n^1\right)$ *does not hold for* $n = k$. *Here* $f_i\left(z\right)$ *and* $u_i\left(z\right)$ *for* $i \neq 1$ *are not transcendental and also* $T\left(r, f_1\right) \neq O^*\left(\left(\log r\right)^{\beta}\right)$ *where* $\beta > 1$ *is a constant.*

References

[1] D. Banerjee and R.K. Dutta, Growth of a class of iterated entire functions, Bull. Math. Anal. Appl. $3\,(1)\,(2011)$, $77-88$.

[2] D. Banerjee and B. Mandal, Relative fix points of a certain class of complex functions, Istanbul Univ. Sci. Fac. J. Math. Phys. Astr. 6 (2015), $15-25$.

[3] D. Banerjee and S. Ghosh, A study on growth of iterated entire functions, accepted in Palestine Journal of Mathematics.

[4] D. Banerjee and S. Ghosh, Study on growth of k-iterated entire functions, South East Asian J. Math. Math. Sci. $16\,(1)$, (2020), $189-206$.

[5] B.K. Lahiri and D. Banerjee, Relative fix points of entire functions, J. Indian Acad. Math.,19 (1) (1997), $87-97$.

[6] K. Niino and N. Suita, Growth of composite function of entire functions, Kodai Math. J. 3 (1980), $374-379$.

[7] J. Sun, Growth of a class of composite entire functions, Acta Math. Vietnamica, 28 (2) (2003), $175-183$.