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# THE CONGRUENCE ON $\Pi^*$-REGULAR SEMIGROUPS

XIAOQIANG LUO

**Abstract:** Let $\rho$ be a congruence on a $\Pi^*$-regular semigroup $S$. Congruence pair $[\rho_T, \rho^T]$, $[\rho_K, \rho^K]$ is constructed by trace and kerner of $S$. To show properties of congruence on $\Pi^*$-regular semigroups using congruence pair. thereby some congruence relation are attained.

## 1. Introduction

The study of the congruence on a semigroup was begon in the 50s.Especially, Howei had given a detailed description of the congruence using the Rees structure theorem. He gaimed many important results by definding two equivalences $\varepsilon_1$ and $\rho_1$. At the same time, many researcher, such as Gluskin, Tamura , Preston , Lallement , Kapp , Schneider , Francis Prestjin and Trotter P.G, also take part in it and get many new results.(see[1-8])

In this paper, using the trace class and kerner class, we describe a classification of the congruence on $\Pi^*$-regular semigroups(see[9]) and discussed the relation among the largest congruence, the smallest congruence and the lattice of congruence on kerner class and trace class.

## 2. Preliminaries

In this section, we gives some basic concepts on the congruence on $\Pi^*$-regular semigroups.

**Definition 1** Let $\rho$ be a congruence on a $\Pi^*$-regular semigroup $S$, $E(S)$ is the set of all idempotents of $S$. $tr\rho$ is called the tace of $\rho$,if

$$tr\rho = \{((e,f),(g,h)) \in E(S) \times E(S), (e,f)\rho(g,h)\}.$$

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**Definiton 2** Let $\rho$ be a congruence on a $\Pi^*$-regular semigroup $S$ . $\ker\rho$ is called the kernel of $\rho$ ,if

$\ker\rho = \bigcup_{(e,f)\in E(S)} \{(e,f)\rho\}$ ,where $(e,f)\rho$ is equivalence class of all idempotents of $S$ .

Suppose $\rho$ is a congruence on a $\Pi^*$-regular semigroup $S$ , then trace class $[\rho_T, \rho^T]$ and kerner class $[\rho_K, \rho^K]$ are existed. Moreover, $\rho$ is decided by $tr\rho$ and $\ker\rho$ ( $\rho_T$ is the least congruence on $S$ with trace $\tau$ , $\rho^T$ is the largest congruence on $S$ with trace $\tau$ , $\rho^K$ is the largest congruence on $S$ with kerner $K$ , $\rho_K$ is the least congruence on $S$ with kerner $K$ ) . $K$ is kerner relation, $T$ is trace relation.

**Remark** The marks we don't illustrate in this paper please see reference ([2],[3],[4]).

**3. Main Results**

**Theorem 1** Let $\rho$ be a congruence on a $\Pi^*$-regular semigroup $S$ . Then $\rho^T \cap \rho^K = \rho$ .

**Proof.** Since $\rho \subseteq \rho^T$ , $\rho \subseteq \rho^K$ (see[5]). Then $\ker\rho \subseteq \ker\rho^T$ , $tr\rho \subseteq tr\rho^T$

On the other hand, $tr\rho \subseteq tr\rho^T$ , $\ker\rho \subseteq \ker\rho^T$ , $tr(\rho^T \cap \rho^K) = tr\rho^T \cap tr\rho^K = tr\rho$

$\ker(\rho^T \cap \rho^K) = \ker\rho^T \cap \ker\rho^K = \ker\rho^T \cap K = K$ . Hence, it follows that

$\rho^T \cap \rho^K = \rho$ .

**Theorem 2** Let $\rho$ be a congruence on a $\Pi^*$-regular semigroup $S$ . Then $\rho = \rho_T \vee \rho_K$ .

**Proof.** Let $\sigma = \rho_T \vee \rho_K$ .Since $\rho_T \subseteq \rho$ , $\rho_K \subseteq K$ ,Then $\sigma = \rho_T \vee \rho_K \subseteq \rho$ ,and so $\rho_T \subseteq \sigma \subseteq \rho$ , $\rho_K \subseteq \sigma \subseteq \rho$ .Such that, we get $tr\sigma \subseteq tr\rho$, $\ker\sigma \subseteq \ker\rho$ .By the definition of $\sigma_T, \sigma_K$ .we have $\sigma_T \subseteq \rho_T$ , $\sigma_K \subseteq \rho_K$ .On the other hand, since $(\rho_T)_T = \rho_T$ , $(\rho_K)_K = \rho_K$ , then $\rho_T \subseteq \sigma_T \subseteq \rho_T$ , $\rho_K \subseteq \sigma_K \subseteq \rho_K$ , and so $\sigma_T = \rho_T$ .We immediately have $\rho = \sigma$ .Hence $\rho = \rho_T \vee \rho_K$ .

**Theorem 3** Let $\rho$ be a congruence on a $\Pi^*$-regular semigroup $S$ . Then

$(\rho_T)_K \vee (\rho_K)_T = \rho_T \wedge \rho_K$.

**Proof.** Since $\rho_T \subseteq \rho$, then $(\rho_T)_K \subseteq \rho_K$, $(\rho_T)_K \subseteq \rho_T$, and so $(\rho_T)_K \subseteq \rho_T \wedge \rho_K$, $(\rho_K)_T \subseteq \rho_T$. Then we have $(\rho_K)_T \vee (\rho_T)_K \subseteq \rho_T \wedge \rho_K$. Let $\sigma = (\rho_K)_T \vee (\rho_T)_K$, $\eta = \rho_T \wedge \rho_K$, then $\sigma \subseteq \eta$. By $\eta \subseteq \rho_T$, $(\rho_T)_K \subseteq \sigma$, we have $(\rho_T)_K \subseteq \sigma \subseteq \eta \subseteq \rho_T$. But we also have $((\rho_T)_K)_K = (\rho_T)_K$, and so $(\rho_T)_K \subseteq \sigma_K \subseteq \eta_K \subseteq (\rho_T)_K$. That is $\sigma_K = \eta_K$. On the other hand, since $\eta \subseteq \rho_K$, $(\rho_K)_T \subseteq \sigma$, Then $(\rho_K)_T \subseteq \sigma \subseteq \eta \subseteq \rho_K$. By $((\rho_K)_T)_T = (\rho_K)_T$, we immediately $(\rho_K)_T \subseteq \sigma_T \subseteq \eta_T \subseteq (\rho_K)_T$. So This prove that $\ker\sigma_T = \ker\eta_K, tr\sigma_T = tr\eta_T$. But for $\ker\sigma_T = \ker\sigma, \ker\eta_K = \ker\eta$, $tr\sigma_T = tr\sigma$, $tr\eta_T = tr\eta$, we have $\ker\sigma = \ker\eta$, $tr\sigma = tr\eta$, then $\sigma = \eta$. Hence $(\rho_T)_K \vee (\rho_K)_T = \rho_T \wedge \rho_K$.

**Theorem 4** Let $\rho$ be a congruence on a $\Pi^*$-regular semigroup $S$. Then $\sigma\eta = \eta\sigma$ for some congruence $\sigma, \eta \in [\rho_T, \rho^T]$.

**Proof.** Suppose that congruence $\sigma, \eta \in [\varepsilon, \varepsilon^T]$ such that $\sigma, \eta \subseteq \varepsilon^T$, where $\varepsilon$ is a equal relation on $S$. Let $H^0$ is $H$-class with 0. Since $\varepsilon^T = H^0 \subseteq H^*$, then we have $\sigma \subseteq H^* \subseteq R^*, \eta \subseteq H^* \subseteq L^*$, Suppose that $(a,b),(c,d) \in S$, such that $(a,b)(\sigma\eta)(c,d)$. Then there exist $(s,t) \in S$ such that $(a,b)\sigma(s,t),(s,t)\eta(c,d)$, and so $(a,b)R^*(s,t),(s,t)L^*(c,d)$. Then it follows from the definition of Green's relation that there exist $(u,v),(i,j),(x,y),(l,r)$ such that (see[10])

$$(s,t)=(a,b)(u,v),(a,b)=(c,d)(i,j),(c,d)=(x,y)(c,d),(l,r)(s,t)=(c,d),$$

And so $(l,r)(a,b)=(l,r)(s,t)(i,j)=(c,d)(i,j)$, By $(s,t)\eta(c,d)$, $(a,b)\sigma(s,t)$, we have $(s,t)(i,j)\eta(c,d)(i,j),(c,d)(i,j)\sigma(c,d)$, and so $(a,b)(\eta\sigma)(c,d)$.

Hence $\sigma\eta \subseteq \eta\sigma$ ; Similarly,we have $\eta\sigma \subseteq \sigma\eta$ .It is clear that $\sigma\eta = \eta\sigma$ .

In the following, we prove that $\sigma\eta = \eta\sigma$ is hold for all $\sigma, \eta \subseteq [\rho_T, \rho^T]$, Since the interval $[\rho_T, \rho^T]$ is isomorphic to the interval $[\varepsilon, \varepsilon^T]$ of the lattice of congruence on $S/\rho_T$ and $\sigma/\rho_T, \eta/\rho_T \in [\varepsilon, \varepsilon^T]$.Then we have

$$(\sigma/\rho_T)(\eta/\rho_T) = (\eta/\rho_T)(\sigma/\rho_T).$$

Suppose that $(a,b),(c,d) \in S$ ,such that $(a,b)(\sigma\eta)(c,d)$. Then there exist $(s,t) \in S$ such that $(a,b)\sigma(s,t),(s,t)\eta(c,d)$, by $\rho_T \subseteq \sigma, \rho_T \subseteq \eta$ , we have

$$((a,b)\rho_T)\sigma/\rho_T((s,t)\rho_T),((s,t)\rho_T)\eta/\rho_T((c,d)\rho_T).$$

From above we know there exist $d \in S$ such that

$$((a,b)\rho_T)\eta/\rho_T((h,g)\rho_T),((h,g)\rho_T)\sigma/\rho_T((c,d)\rho_T),$$

that is $(a,b)\eta(h,g),(h,g)\sigma(c,d)$, and so $(a,b)(\eta\sigma)(c,d)$ .Hence $\sigma\eta \subseteq \eta\sigma$ . Similaryly, we have $\eta\sigma \subseteq \sigma\eta$ . Hence, it fpllow that $\sigma\eta = \eta\sigma$ .

**Theorem 5** Let $\rho$ be a congruence on a $\Pi^*$ -regular semigroup $S$ .Then the interval $[\rho_T, \rho^T]$ is a modular lattice.

**Proof.** Let $\sigma$, $\eta$, $\lambda$ be for some congruence on the interval $[\rho_T, \rho^T]$, and $\sigma \subseteq \lambda$ .Since $(\sigma \vee \eta) \wedge \lambda \supseteq \sigma \vee (\eta \wedge \lambda)$ has been known,in here we only show $(\sigma \vee \eta) \wedge \lambda \subseteq \sigma \vee (\eta \wedge \lambda)$ .At first, $[\rho_T, \rho^T]$ is sublattice ,for some $\sigma$, $\eta$, $\lambda \in [\rho_T, \rho^T]$,this has $\sigma \vee \eta \in [\rho_T, \rho^T]$, $\sigma \vee (\eta \wedge \lambda) \in [\rho_T, \rho^T]$,second , by theorem 4 $\sigma\eta = \eta\sigma$ ,so $\sigma \vee \eta = \sigma\eta$ ; similarly $\sigma(\eta \wedge \lambda) = (\eta \wedge \lambda)\sigma$ ,hence, $\sigma \vee (\eta \vee \lambda) = \sigma(\eta \wedge \lambda)$ .Now we only prove $\sigma\eta \wedge \lambda \subseteq \sigma(\eta \wedge \lambda)$ .Suppose that $(a,b),(c,d) \in S$ ,$(a,b)(\sigma\eta \wedge \lambda)(c,d)$ ,thus $(a,b)(\sigma\eta)(c,d)$ and $(a,b)\lambda(c,d)$, for $(s,t)\in S$ ,such that $(a,b)\sigma(s,t),(s,t)\eta(c,d)$ .Because $\sigma\subseteq\lambda$ ,we have $(a,b)(\sigma(\eta\wedge\eta))(c,d)$ ,so $(\sigma\vee\eta)\wedge\lambda\subseteq\sigma\vee(\eta\wedge\lambda)$ ,hence the interval $[\rho_T,\rho^T]$ is a modular lattice.

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DEPARTMENT OF MATHEMATICS SICHUAN UNIVERSITY OF ARTS AND SCIENCE, DAZHOU, SICHUAN, 635000, CHINA