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DUAL PROPERTIES OF $\omega-$ORDER REVERSING PARTIAL CONTRACTION MAPPING IN SEMIGROUP OF LINEAR OPERATOR

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Abstract. This paper consists of results on dual properties of a $\omega-$order reversing partial contraction mapping (ω-$ORCP_n$), we assume a Banach space $X = H$ (Hilbert space). We established that the operator $A \in \omega$-$ORCP_n$ which is the infinitesimal generator of $C_0 - Semigroup$ is reflexive, densely defined and a closed linear operator which is now considered as a semigroup of linear operator.

1. Introduction and preliminaries

Dual properties of a semigroup of linear operator is an important aspect of C_0-semigroup because of the emphasis on weakly topologies of operator thereby making it to obtain a linear operator called the weak generator of a semigroup $(T(t)^*)_{t\geq 0}$. Suppose X is a Banach space, $X_n \subseteq X$ be a finite set, $(T(t))_{t\geq 0}$ the C_0-semigroup, $\omega - ORCP_n$ be ω-order-reversing partial contraction mapping (semigroup of linear operator) which is an example of C_0-semigroup, $\omega - ORCP_n \subseteq ORCP_n$(Order Reversing Partial Contraction Mapping). let $M_m(\mathbb{N} \cup 0)$ be a matrix, $L(X)$ be a bounded linear operator in X, P_n, a partial transformation semigroup, $\rho(A)$, a resolvent of A, where A is the generator of a semigroup of linear operator and A^* is the adoint of A. This paper will focus on dual properties and results on $\omega - ORCP_n$ in a semigroup of linear operator called C_0-semigroup.

Yosida [13] established and proved some results on differentiability and representation of one-parameter semigroup of linear operators. Feller [6], obtained semigroups of transformation in a weak topologies. Phillips [9] deduced the adjoint of semigroup and Balakrishnan [1]

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introduced an operator calculus for infinitesimal generators of semigroup. Belleni-Morante and Mcbride[3] applied some evolution equations on semigroups. Davies [4], obtained one parameter semigroups and Haraux [7] proved some linear semiroups in Banach space. See [5], Engel and Nagel established one-parameter semigroup for linear evolution equations and in same year characterized some strongly continuous groups of linear operators on a Hilbert space. In [12], Vrabie established and proved some results of C_0-semigroup and its applications. Rauf and Akinyele [10], obtained ω-order-preserving partial contraction mapping and established its properties, also in [11], Rauf *et.al.* established some results of stability and spectra properties on semigroup of linear operator.

Definition 1.1 ($C_0 - Semigroup$) [12]
A C_0-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.

Definition 1.2 (ω-$ORCP_n$) [10]
A transformation $\alpha \in P_n$ is called ω-order-reversing partial contraction mapping if $\forall x, y \in$Domα : $x \leq y \implies \alpha x \geq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.

Definition 1.3 (Dual of Semigroup) [12]
The family $\{T(t)^*; t \geq 0\} \subseteq L(X^*)$, where, for each $t \geq 0$, $T(t)^*$ is the adjoint of the operator $T(t)$, is called the dual of the semigroup$\{T(t)^*; t \geq 0\}$.

Definition 1.4 (Sun Dual Semigroup)[5]
Corresponding to a strongly continuous semigroup $(T(t)^*)_{t\geq 0}$ on a Banach space X, we defined its sun dual (or semigroup dual) by

$X^{\odot} = \{x^* \in X^* : \lim_{t\to\infty} \|T(t)^*x^* - x^*\| = 0\}$ and call the semigroup given by the restricted operators.

$T(t)^{\odot} = T(t)^*$, $t \geq 0$, the sun dual semigroup.

Definition 1.5 (Translation Semigroup)[5]
The (left) translation operators $[T(t)f(s) = f(s+t)]$, $s, t \in \mathbb{R}$, define a strongly continuous (semi)group on the spaces $C_{ub}\mathbb{R}$ and $L^P(\mathbb{R})$, $1 \leq p < \infty$.

Definition 1.6 (Adjoint Semigroup)[12]
Let H be a real Hilbert space identified with its own topological dual. The operator $A : D(A) \subseteq H \to H$ is called:
(i) self-adjoint if $A = A^*$
(ii) skew-adjoint if $A = -A^*$
(iii) symmetric if $< Ax, y >=< x, Ay >$ for each $x, y \in D(A)$;
(iv) skew-symmetric if $< Ax, y >= - < x, Ay >$ for $x, y \in D(A)$.

1.1. **Some Basic Properties of Dual Semigroup Operator.** For a semigroup to be dual it is not necessarily strongly continuous on X^* and it is still possible to associate a

"generator" to it. Hence, it needs to be:
(i) Translative, i.e.
$[T(t)f(s) = f(s+t)]$, $s, t \in \mathbb{R}$.
(ii) Adjoint, i.e.
$A = A^*$
$A = -A^*$
$< Ax, y >=< x, Ay >$
$< Ax, y >= - < x, Ay >$.
(iii) Dual and Sun Dual i.e
for $\{T(t)^*; t \geq 0\} \subseteq L(X^*)$, then we have
$X^{\odot} = \{x^* \in X^* : \lim_{t\to\infty} \|T(t)^*x^* - x^*\| = 0\}$, where
$T(t)^{\odot} = T(t)^*$, $t \geq 0$.

Example 1
2×2 matrix $[M_m(\mathbb{N} \cup \{0\})]$
Suppose

$$A = \begin{pmatrix} 2 & 2 \\ 2 & - \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{2t} & e^{2t} \\ e^{2t} & I \end{pmatrix}.$$

Example 2
3×3 matrix $[M_m(\mathbb{N} \cup \{0\})]$
Suppose

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{3t} & e^{2t} & e^{t} \\ e^{2t} & e^{2t} & e^{t} \\ e^{3t} & e^{2t} & e^{2t} \end{pmatrix}.$$

Example 3
3×3 matrix $[M_m(\mathbb{C})]$, we have
for each $\lambda > 0$ such that $\lambda \in \rho(A)$ where $\rho(A)$ is a resolvent set on X.

Suppose we have

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA_\lambda}$, then

$$e^{tA_\lambda} = \begin{pmatrix} e^{3t\lambda} & e^{2t\lambda} & e^{t\lambda} \\ e^{2t\lambda} & e^{2t\lambda} & e^{t\lambda} \\ e^{3t\lambda} & e^{2t\lambda} & e^{2t\lambda} \end{pmatrix}.$$

Example 4

Let $\{T(t); t \in \mathbb{R}_+\} \subset L(L^1(\mathbb{R}_+)$ be the translation semigroup , i.e.

$$[T(t)f](s) = f(t+s)$$

for each $t \geq 0$, each $f \in L^1(\mathbb{R}_+)$ and for $s \in \mathbb{R}_+$. Recalling that the dual of $L^1(\mathbb{R}_+)$ is L^∞, we easily deduce that, for each $t \in \mathbb{R}_+$, the dual semigroup $T(t)^* : L^\infty(\mathbb{R}_+) \to L^\infty(\mathbb{R}_+)$ is defined by $[T(t)^*\varphi](s) = \varphi(s+t)$ for each $\varphi \in L^\infty(\mathbb{R}_+)$, and $s \in \mathbb{R}_+$. This follows from

$(T(t)f, \varphi) = \int_{\mathbb{R}_+} f(s-t)\varphi(s)ds = \int_{\mathbb{R}_+} \varphi(s)f(s-t)ds$ for each $f \in L^1(\mathbb{R}_+)$, and each $\varphi \in L^\infty(\mathbb{R}_+)$. On the other hand, we can easily deduce that $\{T(t)^*; t \geq 0\}$ does not satisfy the condition $\lim_{t\to 0} T(t)^*\varphi = \varphi$ except if φ is "uniformly continuous" on $\mathbb{R}_+$ with respect to $\|.\|_{L^\infty(\mathbb{R}_+)}$.

1.2. **Theorem.** Hille-Yoshida [12]

A linear operator $A : D(A) \subseteq X \to X$ is the infinitesimal generator for a C_0-semigroup of contraction if and only if

i. A is densely defined and closed,

ii. $(0, +\infty) \subseteq \rho(A)$ and for each $\lambda > 0$, we have

$$\|R(\lambda, A)\|_{L(X)} \leq \frac{1}{\lambda}. \tag{1.1}$$

2. Main results

This section presents the result of approximations on ω-OCP_n in semigroup of linear operator:

Theorem 2.1

Suppose $X = H$ is a complex Hilbert space with inner product $< .,. >$, let $A : D(A) \subseteq H \to H$ be densely defined, $\mathbb{C}$-linear operator and let iA be defined by

$$\begin{cases} D(iA) = D(A) \\ (iA)x = iAx \text{ for each } x \in D(iA) \end{cases}$$

for each $x \in D(iA)$ and $A \in \omega$-$ORCP_n$.

Then A^* is $\mathbb{C}$-linear and $(iA)^* = -iA^*$ and A is self-adjoint if and only if A is skew-adjoint.

Proof

Since A is skew-adoint if and only if iA is self-adjoint, then the adjoint of A on the complex Hilbert space H identified with its own topological dual H_c^*, coincides with the adjoint of A on the real Hilbert space H is identified with its own topological dual H_r^*. Then, let $x \in D(A^*)$ and $A \in \omega\text{-}ORCP_n$. For each $y \in D(A)$ and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} < \lambda A^* x, y > &=< A^* x, \bar{\lambda} y >=< x, A(\bar{\lambda} y) > \\ &=< x, \bar{\lambda} A y > \\ &=< \lambda x, Ay > \\ &=< A^*(\lambda x), y > . \end{aligned} \tag{2.1}$$

Since $D(A)$ is the dense in H and $y \in D(A)$ arbitrary, from (2.1) above its follows that $A^*(\lambda x) = \lambda A^* x$, which proves that A^* is $\mathbb{C}$-linear. Now, let $x \in D(A^*)$ and $y \in D(A)$. Then we have

$$\begin{aligned} < -iA^* x, y > &=< A^* x, iy > \\ &=< x, A(iy) > \\ &=< x, iAy > \\ &=< (iA)^* x, y > . \end{aligned} \tag{2.2}$$

So $x \in D((iA)^*)$, and

$$\text{graph } (-iA^*) \subseteq graph((A)^*), \tag{2.3}$$

substituting A by iA in (2.3), we have

$$\text{graph } (-i(iA)^*) \subseteq \text{graph } ((-A)^*). \tag{2.4}$$

From the $\mathbb{C}$-linearity, we conclude that

$$\text{graph } (iA)^* \subseteq \text{graph } ((-iA)^*). \tag{2.5}$$

and thus it follows that

$$\text{graph } (iA)^* = \text{graph } (-A^*). \tag{2.6}$$

Finally, if A is self-adjoint, from the previous consideration, we conclude that

$$(iA)^* = -iA^* = iA \tag{2.7}$$

Which shows that iA is skew-adjoint. Conversely if iA is skew-adjoint, we have

$$A^* = (-i(iA))^*) = i(iA)^* = -i(iA) = A \tag{2.8}$$

and hence A is self-adjoint. Similarly, if A is skew-adjoint, then

$$(iA)^* = -iA^* = iA \tag{2.9}$$

and so iA is self-adjoint. Conversely, if iA is self-adjoint, then

$$A^* = (-i(iA))^*) = i((iA)^*) = i(iA) = -A \tag{2.10}$$

Hence, the proof is complete.

Theorem 2.2

Let $A \in \omega\text{-}ORCP_n$, where $\omega\text{-}ORCP_n \in L(X)$, then it follows that $A \in L(X)$ and $A^* \in L(X^*)$ so that

$$\|A\|_{L(X)} = \|A^*\|_{L(X^*)}$$

Proof

Suppose $x^* \in X^*$, then $x \to (Ax, x^*)$ is a linear continuous functional on X, denoted by y since

$$(x, y^*) = (Ax, x^*) \tag{2.11}$$

we have

$$D(A^*) = X^* \tag{2.12}$$

In addition

$$\begin{aligned} \|A^*\|_{L(x^*)} = \sup_{\|x^*\|\leq 1} \|A^*x^*\| \\ &= \sup_{\|x\|\leq 1} \sup_{\|x\|\leq 1} |(x, A^*x^*)| \\ &= \sup_{\|x\|\leq 1} \sup_{\|x^*\|\leq 1} |(Tx, x^*)| \\ &= \sup_{\|x\|\leq 1} \|Ax\| \\ &= \|A\|_{L(x)} \end{aligned}$$

Which complete the proof

Theorem 2.3

Let X be reflexive and let $A : D(A) \subseteq X \to X$, where $A \in \omega\text{-}ORCP_n$ (densely defined linear closed operator), that is the infinitesimal generator of a C_0-semigroup of contractions $\{T(t); t \geq 0\}$. Then

i. A^* is densely defined and closed,
ii. $\{T(t); t \geq 0\}$ is a C_0-semigroup of contractions whose infinitesimal generator is $A^* : D(A^*) \subseteq X^* \to X^*$.

Proof

we need to show that A^* is densely defined. By contradiction, assume there exist at least one reflexive Banach space and at least one densely defined linear operator $A : D(A) \subseteq X \to X$

for which $D(A^*)$ is not dense in X^*.
This means that there exists at least one element $y^{**} \in X^{**}$, such that $y^{**} \neq 0$ and

$$(x^*, y^{**}) = 0 \tag{2.13}$$

for each $x \in D(A)$ and $A \in \omega\text{-}ORCP_n$. Since X is reflexive, the statement above is equivalent with : there exists $y \in X$ such that
$y \neq 0$ and $(y, x^*) = 0$,
for each $x^* \in D(A^*)$ and $A \in \omega\text{-}ORCP_n$. Since the graph of A is closed in $X \times X$, it follows that $(0, y) \notin graph(A)$. By a consequence of the Hahn-Banach seperation theorem applied to $graph(A)$ and $(0, y)$ in $X \times X$, its follows that there exists $x_1^*, x_2^* \in X^*$ such that

$$(x, x_1^*) - (Ax, x_2^*) = 0 \tag{2.14}$$

for each $x \in D(A)$, and $(0, x_1^*) - (y, x_2^*) \neq 0$.
From the second relation, it follows that both $x_2^* \neq 0$ and $(y, x_2^*) \neq 0$.
On the other hand, from (2.14), we deduce that $x_2^* \in D(A^*)$ and $A^*x_2^* = x_1^*$, which implies
$(y, x_2^*) = 0$
This contradiction can be eliminated only if $D(A^*)$ is dense in X^*. In order to prove that A^* is closed, let $(x_n^*)_{n\in\mathbb{N}}$ be a sequence in $D(A^*)$ with the property that
$\lim_{n\to\infty} x_n^* = x^*$ and $\lim_{n\to\infty} A^*x_n^* = y^*$.
By definition of the operator A^*, we have
$(Ax, x^*) = \lim_{n\to\infty}(Ax, x_n^*) = \lim_{n\to\infty}(x, A^*x_n^*) = (x, y^*)$
for each $x \in D(A)$ and $A \in \omega\text{-}ORCP_n$. Since $D(A)$ is dense, it follows that $x^* \in D(A^*)$ and

$$A^*x^* = y^* \tag{2.15}$$

So A^* is closed, and that complete the proof of (i).
To prove (ii), we need to show that A^* satisfies the conditions of Theorem 1.2. By (i) above, A^* is densely defined and closed. So A^* satisfies (i) in Theorem 1.2. We prove that $(0, +\infty) \subseteq \rho(A^*)$ and for each $\lambda > 0$, we have

$$\|R(\lambda, A)\|_{L(X^*)} \leq \frac{1}{\lambda} \tag{2.16}$$

Since A is infinitesimal generator of a C_0-semigroup of contractions and $A \in \omega\text{-}ORCP_n$, we have $(0, +\infty) \subseteq \rho(A^*)$.

Suppose there exists $\lambda \in \rho(A)$ and $\lambda \in \rho(A^*)$, then we have

$$R(\lambda; A^*) = R(\lambda; A)^* \tag{2.17}$$

By virtue of (2.17), we have $\rho(A) \subseteq \rho(A^*)$ and thus $(0, +\infty) \subseteq \rho(A^*)$.
By (2.17) and Theorem 1.2, it follows that

$$\|R(\lambda, A)^*\|_{L(X^*)} = \|R(\lambda, A)\|_{L(X)} \tag{2.18}$$

Recalling that

$$\|R(\lambda, A)\|_{L(X)} \leq \frac{1}{\lambda} \tag{2.19}$$

for each $\lambda > 0$, we have

$$\|R(\lambda, A^*)\|_{L(X^*)} \leq \frac{1}{\lambda} \tag{2.20}$$

for each $\lambda > 0$. Hence, A^* is infinitesimal generator of a C_0-semigroup of contractions $\{T^*(t); t \geq 0\}$. To conclude the proof, we need to show that $T^*(t) = T(t)^*$ for each $t \geq 0$. To show this, let us recall that for each $x^* \in D(A^*)$,

$$T^*(t)x^* = \lim_{\lambda\to\infty} e^{t(A^*)\lambda}x^*$$

where $(A^*)_\lambda$ is the Yosida approximation of the operator A^*. By virtue of Theorem 2.2, we have

$$(A^*)_\lambda = (A_\lambda)^* \tag{2.21}$$

In addition

$$e^{t(A_\lambda)^*} = (e^{tA_\lambda})^* \tag{2.22}$$

and thus,

$$\lim_{\lambda\to\infty} e^{t(A^*)\lambda}x^* = \lim_{\lambda\to\infty} e^{t(A_\lambda)^*}x^* = T(t)^*x^* \tag{2.23}$$

for each $x^* \in D(A^*)$ and $A \in \omega\text{-}ORCP_n$. Since $D(A^*)$ is dense in X^*, hence the proof is complete.

Theorem 2.4
Let $\{T(t); t \geq 0\}$ be a C_0-semigroup of contractions on X with the infinitesimal generator of $A \in \omega\text{-}ORCP_n$ and let $\{T(t)^*; t \geq 0\}$ be the dual semigroup. If A^* is the adjoint of A and $X^\odot$ the closure of $D(A^*)$ in X^*, then the restriction $T(t)^\odot$ of $T(t)^*$ to $X^\odot$ is a $C_0 - Semigroup$ of contractions whose infinitesimal generator $A^\odot$ is the part of A^* in $X^\odot$.

Proof:
Let

$$X^\odot = \{x^\odot \in X^*; \lim_{t\to\infty} \|T(t)^*x^\odot - x^\odot\| = 0\}. \tag{2.24}$$

Clearly $X^\odot$ is a subspace in X^* and $D(A^*)$ is dense in $X^\odot$. If $x^\odot \in D(A^*)$, $x \in X$ and $A \in \omega\text{-}ORCP_n$, we have

$$|(x, T(t)^*x^\odot - x^\odot| = |(T(t)x - x, x^\odot)| \tag{2.25}$$

$$= |A\int_0^t T(s)xds, x^{\odot}| \leq t\|x\|\|A^*x^{\odot}\| \tag{2.26}$$

Accordingly, $\lim_{t\to 0}|(x, T(t)^*x^{\odot} - x^{\odot})| = 0$
uniformly for $\|x\| \leq 1$ and therefore

$$\lim_{t\to 0}\|T(t)^*x^{\odot} - x^{\odot}\| = 0 \tag{2.27}$$

. Thus , $D(A^*) \subseteq X^{\odot}$. On the other hand, let us observe that $\|T(t)^{\odot}x^{\odot} - x^{\odot}\|$
$\leq \|T(t)^{\odot}x^{\odot} - T(t)^{\odot}y^{\odot}\| + \|T(t)^{\odot}y^{\odot} - y^{\odot}\| + \|y^{\odot} - x^{\odot}\|$
for each $x^{\odot}$, $y^{\odot} \in X^{\odot}$. Since by Theorem 2.2,

$$\|T(t)^{\odot}\|_{L(X^{\odot})} \leq \|T(t)^*\|_{L(X^*)}\| = \|T(t)\|_{L(X)}\| = 1 \tag{2.28}$$

(2.28) above shows that $X^{\odot}$ is closed and $\{T(t)^{\odot}; t \geq 0\}$ is a $C_0-semigroup$ of contractions on $X^{\odot}$. Let denote by $A^{\odot}$ its infinitesimal generator, and remark that $D(A^{\odot}) \subseteq D(A^*)$. Thus $D(A^*)$ is dense in $X^{\odot}$, $A^{\odot}$ is the part of A^* in $X^{\odot}$ and $T(t)^{\odot}$ is the part of $T(t)^*$ in $X^{\odot}$ and this complete the proof.

Conclusion

In this paper, considering $A \in \omega-ORCP_n$ as the infinitesimal generator of $C_0-Semigroup$, we hereby obtained dual and sun dual properties of a semigroup of linear operator.

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