ASIAN JOURNAL OF MATHEMATICS AND APPLICATIONS
Volume 2021, Article ID ama0573, 5 pages
ISSN 2307-7743
http://scienceasia.asia

ON A SUBCLASS OF λ- PSEUDO STARLIKE FUNCTIONS

O.Y. SAKA-BALOGUN[1,*], K.O. BABALOLA[2]

[1]Department of Mathematical and Physical Sciences, Afe Babalola University, Ado-Ekiti, Nigeria

[2]Department of Mathematics, University of Ilorin, Ilorin. Nigeria

*Correspondence: balogunld@abuad.edu.ng

Abstract: In this paper, we introduced a new subclass of λ-Pseudo starlike univalent functions in the open unit disk E= $\{Z\epsilon\mathbb{C}|z|\leq 1\}$, denoted by $F_\lambda(\beta)$. Coefficient inequalities and Fekete-Szego functional of this class is shown.

1. INTRODUCTION

Let A denote the class of functions

$$f(z) = z + a_2z^2 + a_3z^3 + \ldots.. \qquad (1.1)$$

Which are holomorphic in £ and by S the subclass of A which consist of univalent functions only. Let $S^*(\beta)$ and $k(\beta)$ be subclasses of S consisting of functions which are respectively of, starlike and convex functions of order β, $0 \leq \beta \leq 1$ in £. That is functions satisfying respectively, $\mathrm{Re}\,\frac{zf'(z)}{f(z)} > \beta$ and $\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta$ in £.

Babalola [1] defined a new class $L_\lambda(\beta)$ of λ-pseudo starlike functions of order β as

$$\mathrm{Re}\frac{z(f'(z))^\lambda}{f(z)} > \beta, \qquad z \in E.$$

Definition 1

An analytic function f ∈ A belongs to $F_\lambda(\alpha)$ if and only if it satisfies the geometric condition

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)} + \frac{1}{\lambda}\frac{zf'}{f}[(f'(z))^{\lambda-1} - 1] > \beta\,;\ z \in\ £\right\} \qquad (1.2)$$

where $\lambda \geq 1$ is real and $0 \leq \beta = \frac{\lambda-1}{\lambda} < 1$

The class $F_\lambda(\beta)$ is a subclass of $L_\lambda(\beta)$ which consists of normalized analytic functions satisfying the geometric condition (1.1). This class of function reduces to class of convex function at $\lambda = 1$.

2. PRELIMINARY LEMMAS

In this paper we shall require the following preliminary results:

Key words and phrases: analytic and univalent functions; convex functions; coefficient inequalities.

Let P_β be the class of functions

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

which are analytic in the unit disk E and satisfy $Re\, p(z) > \beta$ there, which means p(z) have positive real part of order β and in the class P(z). Any map p(z) in the class P can be written as p(z) = $(1-\beta)p_1 + \beta$ where $p_1 \in P$. The class of functions P_β is called Caratheodory functions of order β. If $\beta = 0$, then, we write P in place of P_β.

Lemma 2. 1 [4]

We shall need the well known Caratheodory inequality for P, which is $|c_k| \leq 2, k = 1,2,3, \ldots$ and the following coefficient inequalities for P.

Lemma 2.2 [2]

Let $p \in P$, then

$$\left|c_2 - \sigma\frac{c_1^2}{2}\right| \leq \begin{cases} 2(1-\sigma), & if\ \sigma \leq 0, \\ 2, & if\ 0 \leq \sigma \leq 2, \\ 2(\sigma - 1), & if\ \sigma \geq 2. \end{cases}$$

Lemma 2.3 [3]

Let $u = u_1 + u_2 i, v = v_1 + v_2 i$ and $\psi(u, v)$ be complex valued maps satisfying:

(a) In a domain Ω of $\mathbb{C}^2$, the function $\psi(u,v)$ is continuous.

(b) $Re\, \psi(1,0) > 0$ for $(1,0) \in \Omega$

(c) $Re\, \psi(\zeta + (1-\zeta)u_2 i, v_i) \leq \zeta$, when $(\zeta + (1-\zeta)u_2 i, v_i) \in \Omega$ and $2v_1 \leq -(1-\zeta)(1+u_2^2)$ for real $0 \leq \zeta < 1$.

If $p \in P$ such that $(p(z),\ zp'(z)) \in \Omega$ and $Re\ (p(z),\ zp'(z)) > \zeta$ for $z \in E$, then $Re\, p(z) > \zeta$ in E

3. MAIN RESULTS

In this section, sufficient inclusion, the bounds on the second and third coefficients of functions in the class F_λ are obtained. Also, estimates of the Fekete-Szego functional of this class of function is determined.

Theorem 1

$$F_\lambda \subset L_\lambda(\beta)$$

Let f $\in F_\lambda$,then for some p $\in$ P, we have

$$p(z) = \frac{z(f'(z))^\lambda}{f(z)}$$

Then logarithmic differentiation yields

$$p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{z(f'(z))^\lambda}{f(z)} - \frac{zf'(z)}{f(z)} + \frac{\lambda z f''(z)}{f'(z)}$$

$$= \lambda\left[\ \frac{1-\lambda}{\lambda} \ + \ 1+\frac{zf''(z)}{f'(z)} + \frac{1}{\lambda}\ \frac{z\,f'(z)}{f(z)}\left(f'^{(\lambda-1)}(z)-1\right)\right]$$

Since $f \in F_\lambda$, then

$$\text{Re}\ \left[1\ +\ \frac{zf''(z)}{f'(z)}\ +\ \frac{1}{\lambda}\ \frac{z\,f'(z)}{f(z)}\ \left(f'^{(\lambda-1)}(z)\ -1\ \right)\ \right] \geq \frac{\lambda-1}{\lambda} = \beta$$

which implies

$$\text{Re}\ \ \left(p(z)\ +\ \frac{zp'(z)}{p(z)}\right) > \beta$$

Where $0 \leq \beta = \frac{\lambda-1}{\lambda} < 1$.

Define

$$\psi(u,v) = u + \frac{v}{u},$$

Clearly $\psi(u,v)$ satisfies conditions (a) and (b) of Lemma 2.3. Also

$$Re\ \psi(\beta+(1-\beta)u_2 i, v_1) = \beta + \frac{\beta v_1}{\beta^2+(1-\beta)^2 u_2^2} < \beta$$

Whenever

$$v_1 \leq \frac{-(1-\beta)(1+u_2^2)}{2}$$

Therefore ψ satisfies all the conditions of the Lemma 2.3 and so $Re\ p(z) > \beta$

Hence, $f \in L_\lambda$. And so

$$F_\lambda \subset L_\lambda(\beta)$$

This implies that the class F_λ is contained in $L_\lambda(\beta)$, a class studied by Babalola [1], which has been shown to consist of univalent functions only in E.

Theorem 2

\`Let $f \in F_\lambda(\beta)$. Then

$$|a_2| \leq \frac{2}{4\lambda-2}$$

$$|a_3| \leq \begin{cases} \dfrac{24\lambda^2}{(4\lambda-2)^2(9\lambda-3)} & if\ 1 \leq \lambda \leq 2+\sqrt{3} \\ \dfrac{2}{9\lambda-3} & if\ \lambda \geq 2+\sqrt{3} \end{cases}$$

Proof

For $f \in F_\lambda$, there exists p$\in P$ such that

$$\text{Re}\left[1+\frac{zf''(z)}{f'(z)}+\frac{1}{\lambda}\frac{zf'(z)}{f(z)}\ \ [(f'(z))^{(\lambda-1)}-1]\right] > \frac{\lambda-1}{\lambda}$$

We have

$$\text{Re}\left[1+\frac{zf''(z)}{f'(z)}+\frac{1}{\lambda}\frac{zf'(z)}{f(z)}\ \ [(f'(z))^{(\lambda-1)}-1\right] > \frac{\lambda-1}{\lambda} = \beta$$

For p $\in P(z)$

$$\left[1+\frac{zf''(z)}{f'(z)}+\frac{1}{\lambda}\frac{zf'(z)}{f(z)}\ \left[(f'\ (z))^{(\lambda-1)}-1\right]>\beta+(1-\beta)p(z)$$

Where $\beta=\frac{\lambda-1}{\lambda}$. If we multiply both sides of the above equation by $\lambda f(z)f'(z)$, we have

$\lambda f(z)f'(z)+\lambda f(z)f''(z)+f'(z)z(f'(z))^{\lambda}-z(f'(z))^2=\ \lambda f(z)f'(z)[\beta+(1-\beta)p(z)]$

Expanding in series forms, we have

$\lambda z+z^2[7a_2\lambda-2a_2]+z^3[13a_3\lambda+6a_2^2\lambda+2a_2^2\lambda^2-3a_3-4a_2^2]+\cdots=\lambda z+z^2[c_1\lambda(1-\beta)+3a_2\lambda]+z^3[c_2\lambda(1-\beta)+3a_2c_1\lambda(1-\beta)+4a_3\lambda+2a_2^2\lambda]+\cdots$

Comparing coefficients, we have

$$2(2\lambda-1)a_2=c_1$$

By using the caratheodory inequality $|c_1|\leq 2$, gives the bound on a_2, that is

$$|a_2|\leq\frac{2}{4\lambda-2}$$

Next we have

$$a_3(9\lambda-3)=c_2+3a_2c_1+a_2^2(4-4\lambda-2\lambda^2)$$

Which gives

$$(9\lambda-3)|a_3|\leq\left|c_2-\left(\frac{4(\lambda^2-4\lambda+1)}{(16\lambda^2-16\lambda+4)}\right)\frac{c_1^2}{2}\right|$$

Applying Lemma 2.2, with $\sigma=\frac{4(\lambda^2-4\lambda+1)}{(16\lambda^2-16\lambda+4)}$, we have the inequalities for a_3, that is

$$|a_3|\leq\begin{cases}\frac{24\lambda^2}{(4\lambda-2)^2(9\lambda-3)} & if\ 1\leq\lambda\leq 2+\sqrt{3}\\ \frac{2}{9\lambda-3} & if\ \lambda\geq 2+\sqrt{3}\end{cases}$$

Corollary 1

For the case $\lambda=1$, we have

$|a_2|\leq 1, |a_3|\leq 1$

Which are known bounds for the class of convex functrions.

Theorem 3

Let $f\in F_\lambda$. Then

$$|a_3-\mu a_2^2|\leq\begin{cases}\frac{4(6\lambda^2-\mu(9\lambda-3))}{(4\lambda-2)^2(9\lambda-3)} & if\ \mu\leq\frac{8\lambda-2\lambda^2-2}{9\lambda-3}\\ \frac{2}{9\lambda-3} & if\ \frac{8\lambda-2\lambda^2-2}{9\lambda-3}\leq\mu\leq\frac{2(7\lambda^2-4\lambda+1)}{9\lambda-3}\\ \frac{4(\mu(9\lambda-3)-6\lambda^2)}{(4\lambda-2)^2(9\lambda-3)} & if\ \mu\geq\frac{2(7\lambda^2-4\lambda+1)}{9\lambda-3}\end{cases}$$

From (5) and (6),

$$|a_3-\mu a_2^2|\leq\left|\frac{1}{9\lambda-3}\left(c_2-\left(\frac{4(\lambda^2-4\lambda+1)}{(16\lambda^2-16\lambda+4)}\right)\frac{c_1^2}{2}\right)-\mu\left(\frac{c_1}{2(2\lambda-1)}\right)^2\right|$$

Which gives

$$|a_3 - \mu a_2^2| \leq \left|\frac{1}{9\lambda - 3}\left(c_2 - \left(\frac{2[2(\lambda^2 - 4\lambda + 1) + \mu(9\lambda - 3)]}{(4\lambda - 2)^2}\right)\frac{c_1^2}{2}\right)\right|$$

The conclusion follows by taking $\sigma = \frac{2[2(\lambda^2 - 4\lambda + 1) + \mu(9\lambda - 3)]}{(4\lambda - 2)^2}$ in Lemma 2.2.

References

[1] K. O. Babalola, On λ -Pseudo-Starlike functions. Journal of Classical Analysis, 3(2) (2013),137-147.

[2] K. O. Babalola, T. O. Opoola, On the Coefficients of a Certain Class of Analytic Functions. Advances in Inequalities for Series. Nova Science Publishers Inc., New York, (2008), 1-13.

[3] K. O. Babalola, T. O. Opoola, Iterated integral transforms of Caratheodory functions and their applications to analytic and univalent functions. Tamkang Journal of Mathematics, 37(4) (2006), 355-366.

[4] P. L. Duren, Univalent Functions. Springer Verlag. New York, (1983).