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MAXIMUM MODULUS AND MAXIMUM TERM OF GENERALIZED ITERATION OF n ENTIRE FUNCTIONS

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Abstract. In this paper we consider the generalized iteration of n entire functions and compare maximum modulus and maximum term of generalized iterated entire functions with that of the n entire functions.

1. Introduction and Definitions

For an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, let $M(r, f) = \max_{|z|=r} |f(z)|$ and $\mu(r, f) = \max_n |a_n| r^n$ are respectively called maximum modulus and maximum term of $f(z)$ on $|z| = r$. In 1997, Lahiri and Banerjee [7] considered two entire functions $f(z)$ and $g(z)$ and formed the relative iterations of $f(z)$ with respect to $g(z)$ as follows.
$f_1(z) = f(z)$
$f_2(z) = f(g(z)) = f(g_1(z))$
$f_3(z) = f(g(f(z))) = f(g_2(z)) = f(g(f_1(z)))$
... ...
$f_n(z) = f(g(f...(f(z) \text{ or } g(z))...))$
according as n is odd or even
and so are $g_n(z)$.
With this definition of iteration, several researchers (see for example [2], [3], [4]) made close investigation on growth properties of maximum modulus and maximum term of iterated entire functions and achieved various results.
After this in 2012, Banerjee and Mondal [1] introduced a more general type of iteration, called generalized iteration as follows.
Let f and g be two nonconstant entire functions and α be any real number satisfying $0 < \alpha \leq 1$. Then the generalized iteration of f with respect to g is defined as follows.

Key words and phrases. Entire functions, Maximum modulus, Maximum term, Generalized iteration.

$f_{1,g}(z) = (1-\alpha)z + \alpha f(z)$
$f_{2,g}(z) = (1-\alpha)g_{1,f}(z) + \alpha f(g_{1,f}(z))$
$f_{3,g}(z) = (1-\alpha)g_{2,f}(z) + \alpha f(g_{2,f}(z))$
.
.
.
$f_{n,g}(z) = (1-\alpha)g_{n-1,f}(z) + \alpha f(g_{n-1,f}(z))$
and so are
$g_{1,f}(z) = (1-\alpha)z + \alpha g(z)$
$g_{2,f}(z) = (1-\alpha)f_{1,g}(z) + \alpha g(f_{1,g}(z))$
$g_{3,f}(z) = (1-\alpha)f_{2,g}(z) + \alpha g(f_{2,g}(z))$
.
.
.
$g_{n,f}(z) = (1-\alpha)f_{n-1,g}(z) + \alpha g(f_{n-1,g}(z))$.
Recently Banerjee and Sarkar [5] considered n entire functions $f_1(z), f_2(z), ..., f_n(z)$ and defined the relative iteration of n entire functions as follows.
$F_1(z) = f_1(z)$
$F_2(z) = f_2(f_1(z)) = f_2(F_1(z))$
... ...
$F_n(z) = f_n(f_{n-1}(...(f_2(f_1(z))))) = f_n(F_{n-1}(z)),\ n \geq 2.$
Now we introduce a more general type of iteration, called generalized iteration of n entire functions as follows.
Let f_1, f_2,..., f_n are n entire functions and α be any real number satisfying $0 < \alpha \leq 1$. Then we define
$F_1(z) = (1-\alpha)z + \alpha f_1(z)$
$F_2(z) = (1-\alpha)F_1(z) + \alpha f_2(F_1(z))$
$F_3(z) = (1-\alpha)F_2(z) + \alpha f_3(F_2(z))$
.
.
.
$F_n(z) = (1-\alpha)F_{n-1}(z) + \alpha f_n(F_{n-1}(z))$.
Note 1.1. For $\alpha = 1$, generalized iteration reduces to relative iteration of n entire functions.
Following Sato [8], we write $log^{[0]}x = x$, $exp^{[0]}x = x$ and for positive integer m, $log^{[m]}x = log(log^{[m-1]}x)$, $exp^{[m]}x = exp(exp^{[m-1]}x)$.

First we need the following definitions.

Definition 1.1. The order ρ_f and the lower order λ_f of an entire function f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{log^{[2]}M(r,f)}{logr}$$

$$\text{and} \qquad \lambda_f = \liminf_{r\to\infty} \frac{log^{[2]}M(r,f)}{logr}.$$

Singh [9] proved the following relation between $M(r,f)$ and $\mu(r,f)$ as follows.
For $0 \leq r < R$

$$\mu(r,f) \leq M(r,f) \leq \frac{R}{R-r}\mu(R,f).$$

Then one can easily obtain

$$\rho_f = \limsup_{r\to\infty} \frac{log^{[2]}\mu(r,f)}{logr}$$

and

$$\lambda_f = \liminf_{r\to\infty} \frac{log^{[2]}\mu(r,f)}{logr}.$$

The main purpose of this paper is to compare the maximum modulus and maximum term of generalized iterated entire functions with that of the generating functions.

2. Known Results

During the proof of our main results we shall need the following lemmas.

Lemma 2.1. *[6] Let $f(z)$ and $g(z)$ be entire functions with $g(0) = 0$. Let α satisfy $0 < \alpha < 1$ and let $C(\alpha) = \frac{(1-\alpha)^2}{4\alpha}$. Then for $r > 0$*

$$M(r, f\circ g) \geq M(C(\alpha)M(\alpha r, g), f).$$

Further if $g(z)$ is any entire function, then with $\alpha = 1/2$, for sufficiently large values of r

$$M(r, f\circ g) \geq M(\frac{1}{8}M(\frac{r}{2}, g) - |g(0)|, f).$$

Clearly

$$M(r, f\circ g) \geq M(\frac{1}{16}M(\frac{r}{2}, g), f). \tag{2.1}$$

On the other hand from the definition we have

$$M(r, f\circ g) \leq M(M(r,g)), f). \tag{2.2}$$

Lemma 2.2. *[9] Let $f(z)$ and $g(z)$ be entire functions with $g(0) = 0$. Let α satisfy $0 < \alpha < 1$ and let $C(\alpha) = \frac{(1-\alpha)^2}{4\alpha}$. Also let $0 < \delta < 1$. Then*

$$\mu(r, f \circ g) \geq (1 - \delta)\mu(C(\alpha)\mu(\alpha\delta r, g), f).$$

And if $g(z)$ is any entire function, then with $\alpha = \delta = 1/2$, for sufficiently large values of r

$$\mu(r, f \circ g) \geq \frac{1}{2}\mu(\frac{1}{8}\mu(\frac{r}{4}, g) - |g(0)|, f).$$

Clearly

$$\mu(r, f \circ g) \geq \frac{1}{2}\mu(\frac{1}{16}\mu(\frac{r}{4}, g), f). \tag{2.3}$$

Lemma 2.3. *[9] Let $f(z)$ and $g(z)$ be any two entire functions. Then for every $\alpha > 1$ and $0 < r < R$,*

$$\mu(r, f \circ g) \leq \frac{\alpha}{\alpha - 1}\mu(\frac{\alpha R}{R - r}\mu(R, g), f).$$

Clearly for $\alpha = 2$ and $R = 2r$

$$\mu(r, f \circ g) \leq 2\mu(4\mu(2r, g), f). \tag{2.4}$$

3. Main Results

In this section we present the main results of the paper.

Theorem 3.1. *Let $f_1, f_2, ..., f_n$ are n entire functions having positive lower orders and of finite orders and suppose $e^{\gamma(M(\frac{r}{2}, F_n))^\delta} \geq M(r, F_n)$ holds for every $\gamma > 0,\ \delta > 0$. Then*

$$\lim_{r\to\infty} \frac{log^{[2]} M(r, F_n)}{log^{[2]} M(r^A, f_k)} = \infty \tag{3.1}$$

for every positive constant A and $1 \leq k \leq n$.

Proof. Let us suppose that $0 < \alpha < 1$. Choose $0 < \epsilon < min\{\lambda(f_i),\ i = 1\ to\ n\}$. Now for all sufficiently large values of r, using (2.1) we get

$$\begin{aligned} M(r, F_n) &= M(r, (1 - \alpha)F_{n-1} + \alpha f_n(F_{n-1})) \\ &\geq M(r, \alpha f_n(F_{n-1})) - M(r, (1 - \alpha)F_{n-1}) \\ &\geq \alpha M(\frac{1}{16}M(\frac{r}{2}, F_{n-1}), f_n) - (1 - \alpha)M(r, F_{n-1}). \end{aligned}$$

So for all sufficiently large values of r we get

$$\begin{aligned}
log^{[2]}M(r,F_n) &\geq log^{[2]}M(\frac{1}{16}M(\frac{r}{2},F_{n-1}),f_n) - log^{[2]}M(r,F_{n-1}) + O(1) \\
&> (\lambda(f_n)-\epsilon)log(\frac{1}{16}M(\frac{r}{2},F_{n-1})) - log^{[2]}M(r,F_{n-1}) + O(1) \\
&> (\lambda(f_n)-\epsilon)logM(\frac{r}{2},F_{n-1}) - \frac{1}{2}(\lambda(f_n)-\epsilon)logM(\frac{r}{2},F_{n-1}) + O(1) \\
&= \frac{1}{2}(\lambda(f_n)-\epsilon)logM(\frac{r}{2},F_{n-1}) + O(1) \\
&\geq \frac{1}{2}(\lambda(f_n)-\epsilon)log^{[2]}M(\frac{r}{2},F_{n-1}) + O(1) \\
&> \frac{1}{2^2}(\lambda(f_n)-\epsilon)(\lambda(f_{n-1})-\epsilon)log^{[2]}M(\frac{r}{2^2},F_{n-2}) + O(1).
\end{aligned}$$

Repeating the process, after $(n-2)$ steps we get,

$$\begin{aligned}
log^{[2]}M(r,F_n) &> \frac{1}{2^{n-2}}(\lambda(f_n)-\epsilon)(\lambda(f_{n-1})-\epsilon)...(\lambda(f_3)-\epsilon)log^{[2]}M(\frac{r}{2^{n-2}},F_2) + O(1) \\
&> \frac{1}{2^{n-1}}(\lambda(f_n)-\epsilon)(\lambda(f_{n-1})-\epsilon)...(\lambda(f_3)-\epsilon)(\lambda(f_2)-\epsilon)logM(\frac{r}{2^{n-1}},F_1) + O(1) \\
&= \frac{1}{2^{n-1}}(\lambda(f_n)-\epsilon)(\lambda(f_{n-1})-\epsilon)...(\lambda(f_2)-\epsilon)logM(\frac{r}{2^{n-1}},(1-\alpha)z+\alpha f_1) + O(1) \\
&\geq \frac{1}{2^{n-1}}(\lambda(f_n)-\epsilon)...(\lambda(f_2)-\epsilon)[logM(\frac{r}{2^{n-1}},\alpha f_1) - logM(\frac{r}{2^{n-1}},(1-\alpha)z)] + O(1) \\
&= \frac{1}{2^{n-1}}(\lambda(f_n)-\epsilon)(\lambda(f_{n-1})-\epsilon)...(\lambda(f_2)-\epsilon)[logM(\frac{r}{2^{n-1}},f_1) - logM(\frac{r}{2^{n-1}},z] + O(1)
\end{aligned}$$

$$\geq \frac{1}{2^{n-1}}(\lambda(f_n)-\epsilon)(\lambda(f_{n-1})-\epsilon)...(\lambda(f_2)-\epsilon)[(\frac{r}{2^{n-1}})^{\lambda(f_1)-\epsilon} - log\frac{r}{2^{n-1}}] + O(1). \tag{3.2}$$

Now it is possible to choose r sufficiently large so that for every $A>0$

$$log^{[2]}M(r^A,f_k) < (\rho(f_k)+\epsilon)\ logr^A. \tag{3.3}$$

Now from (3.4) and (3.5) we get for sufficiently large values of r,

$$\frac{log^{[2]}M(r,F_n)}{log^{[2]}M(r^A,f_k)} > \frac{\frac{1}{2^{n-1}}(\lambda(f_n)-\epsilon)(\lambda(f_{n-1})-\epsilon)...(\lambda(f_2)-\epsilon)[(\frac{r}{2^{n-1}})^{\lambda(f_1)-\epsilon} - log\frac{r}{2^{n-1}}] + O(1)}{A(\rho(f_k)+\epsilon)\ logr}$$
$$\to \infty\ as\ r\to\infty.$$

Therefore,

$$\lim_{r\to\infty}\frac{log^{[2]}M(r,F_n)}{log^{[2]}M(r^A,f_k)} = \infty.$$

So the result (3.1) is proved.

Theorem 3.2. *Let $f_1, f_2, ..., f_n$ are n non-constant entire functions of finite orders with $\rho(f_1) < \rho(f_n)$. Then*

$$\liminf_{r\to\infty} \frac{log^{[n]}M(r, F_n)}{log^{[2]}M(exp(r^{\rho(f_n)}), f_n)} = 0.$$

Proof. We choose ϵ, so that $0 < \epsilon < \rho(f_n) - \rho(f_1)$.
Since $\rho(f_n) > \rho(f_1) \geq 0$, so that f_n must not be a polynomial. Hence

$$M(r, f_n) \geq r \tag{3.4}$$

for all large values of r.
Now for all large values of r, using (2.2) and (3.4) we obtained that

$$\begin{aligned} M(r, F_n) &\leq (1-\alpha)M(r, F_{n-1}) + \alpha M(r, f_n(F_{n-1})) \\ &\leq (1-\alpha)M(M(r, F_{n-1}), f_n) + \alpha M(M(r, F_{n-1}), f_n) \\ &= M(M(r, F_{n-1}), f_n). \end{aligned}$$

Therefore,

$$log^{[2]}M(r, F_n) < (\rho(f_n) + \epsilon)logM(r, F_{n-1}).$$

So,

$$log^{[3]}M(r, F_n) < (\rho(f_{n-1}) + \epsilon)logM(r, F_{n-2}) + O(1).$$

Therefore,

$$log^{[4]}M(r, F_n) < (\rho(f_{n-2}) + \epsilon)logM(r, F_{n-3}) + O(1).$$

After $(n-2)$ steps we get

$$\begin{aligned} log^{[n]}M(r, F_n) &< (\rho(f_2) + \epsilon)logM(r, F_1) + O(1) \\ &= (\rho(f_2) + \epsilon)logM(r, (1-\alpha)z + \alpha f_1) + O(1) \\ &\leq (\rho(f_2) + \epsilon)[logM(r, \alpha f_1) + logM(r, (1-\alpha)z)] + O(1) \\ &= (\rho(f_2) + \epsilon)[logM(r, f_1) + logM(r, z)] + O(1) \\ &= (\rho(f_2) + \epsilon)[logM(r, f_1) + logr)] + O(1) \\ &\leq (\rho(f_2) + \epsilon)[logM(r, f_1) + logM(r, f_1)] + O(1) \\ &= 2(\rho(f_2) + \epsilon)logM(r, f_1) + O(1) \\ &< (\rho(f_2) + \epsilon)r^{(\rho(f_1)+\epsilon)} + O(1). \end{aligned}$$

On the other hand, for a sequence $r = r_n \to \infty$

$$log^{[2]}M(r, f_n) > (\rho(f_n) - \epsilon)logr.$$

Expressing $R_n = (log\ r_n)^{\frac{1}{\rho(f_n)}}$ it follows that

$$log^{[2]}M(exp(R_n^{\rho(f_n)}), f_n) > (\rho(f_n) - \epsilon){R_n}^{\rho(f_n)}.$$

Thus for $r = R_n (\geq r_0)$

$$\frac{log^{[n]} M(r, F_n)}{log^{[2]} M(exp(r^{\rho(f_n)}), f_n)} < \frac{(\rho(f_2) + \epsilon) r^{(\rho(f_1)+\epsilon)} + O(1)}{(\rho(f_n) - \epsilon) r^{\rho(f_n)}}.$$

Hence,

$$\liminf_{r \to \infty} \frac{log^{[n]} M(r, F_n)}{log^{[2]} M(exp(r^{\rho(f_n)}), f_n)} = 0.$$

Theorem 3.3. *Let $f_1, f_2, ..., f_n$ are n nonconstant entire functions of finite orders with $\lambda(f_1) > \rho(f_k)(1 \leq k \leq n)$ and $\lambda(f_n) > 0$ and suppose $e^{\gamma(M(\frac{r}{2}, F_n))^\delta} \geq M(r, F_n)$ holds for every $\gamma > 0,\ \delta > 0$. Then*

$$\lim_{r \to \infty} \frac{log^{[2]} M(r, F_n)}{log^{[2]} M(exp(r^{\rho(f_k)}), f_k)} = \infty.$$

Proof. We choose ϵ, so that $0 < \epsilon < \lambda(f_1) - \rho(f_k)$. From (3.2) we get for all $r \geq r_0$

$$log^{[2]} M(r, F_n) > \frac{1}{2^{n-1}} (\lambda(f_n) - \epsilon)(\lambda(f_{n-1}) - \epsilon)...(\lambda(f_2) - \epsilon)[(\frac{r}{2^{n-1}})^{\lambda(f_1)-\epsilon} - log \frac{r}{2^{n-1}}] + O(1).$$

On the other hand for all $r \geq r_0$

$$log^{[2]} M(r, f_k) < (\rho(f_k) + \epsilon)\ log\ r$$

$$i.e.,\ log^{[2]} M(exp(r^{\rho(f_k)}), f_k) < (\rho(f_k) + \epsilon)\ r^{\rho(f_k)}.$$

Thus for all sufficiently large r

$$\frac{log^{[2]} M(r, F_n)}{log^{[2]} M(exp(r^{\rho(f_k)}), f_k)} > \frac{\frac{1}{2^{n-1}} (\lambda(f_n) - \epsilon)(\lambda(f_{n-1}) - \epsilon)...(\lambda(f_2) - \epsilon)[(\frac{r}{2^{n-1}})^{\lambda(f_1)-\epsilon} - log \frac{r}{2^{n-1}}] + O(1)}{(\rho(f_k) + \epsilon)\ r^{\rho(f_k)}}$$

$$\to \infty\ as\ r \to \infty.$$

Hence

$$\lim_{r \to \infty} \frac{log^{[2]} M(r, F_n)}{log^{[2]} M(exp(r^{\rho(f_k)}), f_k)} = \infty.$$

Theorem 3.4. *Let $f_1, f_2, ..., f_n$ are n nonconstant entire functions of positive lower orders and of finite orders and suppose $e^{\gamma(\mu(\frac{r}{4}, F_n))^\delta} \geq \mu(r, F_n)$ holds for every $\gamma > 0,\ \delta > 0$ and also for every positive integer n. Then*

$$\lim_{r \to \infty} \frac{log^{[2]} \mu(r, F_n)}{log^{[2]} \mu(r^A, f_k)} = \infty$$

for every positive constant A and $1 \leq k \leq n$.

Proof. We choose ϵ such that $0 < \epsilon < min\{\lambda(f_i),\ i = 1\ to\ n\}$. Now for all sufficiently large values of r, using (2.3) we get

$$\begin{aligned} \mu(r, F_n) &= \mu(r, (1-\alpha) F_{n-1} + \alpha f_n(F_{n-1})) \\ &\geq \mu(r, \alpha f_n(F_{n-1})) - \mu(r, (1-\alpha) F_{n-1}) \\ &\geq \frac{1}{2} \alpha \mu(\frac{1}{16} \mu(\frac{r}{4}, F_{n-1}), f_n) - (1-\alpha) \mu(r, F_{n-1}). \end{aligned}$$

So for all sufficiently large values of r we get

$$\begin{aligned} log^{[2]}\mu(r,F_n) &\geq log^{[2]}\mu(\frac{1}{16}\mu(\frac{r}{4},F_{n-1}),f_n) - log^{[2]}\mu(r,F_{n-1}) + O(1) \\ &> (\lambda(f_n)-\epsilon)log(\frac{1}{16}\mu(\frac{r}{4},F_{n-1})) - log^{[2]}\mu(r,F_{n-1}) + O(1) \\ &> (\lambda(f_n)-\epsilon)log\mu(\frac{r}{4},F_{n-1}) - \frac{1}{2}(\lambda(f_n)-\epsilon)log\mu(\frac{r}{4},F_{n-1}) + O(1) \\ &= \frac{1}{2}(\lambda(f_n)-\epsilon)log\mu(\frac{r}{4},F_{n-1}) + O(1) \\ &\geq \frac{1}{2}(\lambda(f_n)-\epsilon)log^{[2]}\mu(\frac{r}{4},F_{n-1}) + O(1) \\ &> \frac{1}{2^2}(\lambda(f_n)-\epsilon)(\lambda(f_{n-1})-\epsilon)log^{[2]}\mu(\frac{r}{4^2},F_{n-2}) + O(1). \end{aligned}$$

Repeating the process, after $(n-2)$ steps we get

$$\begin{aligned} & log^{[2]}\mu(r,F_n) > \frac{1}{2^{n-2}}(\lambda(f_n)-\epsilon)(\lambda(f_{n-1})-\epsilon)...(\lambda(f_3)-\epsilon)log^{[2]}\mu(\frac{r}{4^{n-2}},F_2) + O(1) \\ & > \frac{1}{2^{n-1}}(\lambda(f_n)-\epsilon)(\lambda(f_{n-1})-\epsilon)...(\lambda(f_3)-\epsilon)(\lambda(f_2)-\epsilon)log\mu(\frac{r}{4^{n-1}},F_1) + O(1) \\ & = \frac{1}{2^{n-1}}(\lambda(f_n)-\epsilon)(\lambda(f_{n-1})-\epsilon)...(\lambda(f_2)-\epsilon)log\mu(\frac{r}{4^{n-1}},(1-\alpha)z+\alpha f_1) + O(1) \\ & \geq \frac{1}{2^{n-1}}(\lambda(f_n)-\epsilon)(\lambda(f_{n-1})-\epsilon)...(\lambda(f_2)-\epsilon)[log\mu(\frac{r}{4^{n-1}},\alpha f_1) - log\mu(\frac{r}{4^{n-1}},(1-\alpha)z)] + O(1) \\ & = \frac{1}{2^{n-1}}(\lambda(f_n)-\epsilon)(\lambda(f_{n-1})-\epsilon)...(\lambda(f_2)-\epsilon)[log\mu(\frac{r}{4^{n-1}},f_1) - log\mu(\frac{r}{4^{n-1}},z] + O(1) \end{aligned}$$

$$\geq \frac{1}{2^{n-1}}(\lambda(f_n)-\epsilon)(\lambda(f_{n-1})-\epsilon)...(\lambda(f_2)-\epsilon)[(\frac{r}{4^{n-1}})^{\lambda(f_1)-\epsilon} - log\frac{r}{4^{n-1}}] + O(1). \tag{3.5}$$

Now it is possible to choose r sufficiently large so that for every $A>0$

$$log^{[2]}\mu(r^A,f_k) < (\rho(f_k)+\epsilon)\ logr^A. \tag{3.6}$$

Now from (3.5) and (3.6) we get for sufficiently large values of r,

$$\begin{aligned} \frac{log^{[2]}\mu(r,F_n)}{log^{[2]}\mu(r^A,f_k)} &> \frac{\frac{1}{2^{n-1}}(\lambda(f_n)-\epsilon)(\lambda(f_{n-1})-\epsilon)...(\lambda(f_2)-\epsilon)[(\frac{r}{4^{n-1}})^{\lambda(f_1)-\epsilon} - log\frac{r}{4^{n-1}}] + O(1)}{A(\rho(f_k)+\epsilon)\ logr} \\ &\to \infty\ as\ r\to\infty. \end{aligned}$$

Hence,

$$\lim_{r\to\infty}\frac{log^{[2]}\mu(r,F_n)}{log^{[2]}\mu(r^A,f_k)} = \infty.$$

This proves the theorem.

Theorem 3.5. *Let $f_1, f_2, ..., f_n$ are n non-constant entire functions of finite orders with $\rho(f_1) < \rho(f_n)$. Then*

$$\liminf_{r\to\infty} \frac{log^{[n]}\mu(r, F_n)}{log^{[2]}\mu(exp(r^{\rho(f_n)}), f_n)} = 0.$$

Proof. We choose ϵ, so that $0 < \epsilon < \rho(f_n) - \rho(f_1)$.
Since $\rho(f_n) > \rho(f_1) \geq 0$, so that f_n must not be a polynomial. Hence

$$r \leq \mu(r, f_n) \leq 2\mu(r, f_n) \tag{3.7}$$

for all large values of r.
Now for all large values of r, using (2.4) and (3.7) we obtained that

$$\begin{aligned}
\mu(r, F_n) &\leq (1-\alpha)\mu(r, F_{n-1}) + \alpha\mu(r, f_n(F_{n-1})) \\
&< (1-\alpha)4\mu(2r, F_{n-1}) + \alpha\mu(r, f_n(F_{n-1})) \\
&\leq (1-\alpha)2\mu(4\mu(2r, F_{n-1}), f_n) + \alpha 2\mu(4\mu(2r, F_{n-1}), f_n) \\
&= 2\mu(4\mu(2r, F_{n-1}), f_n).
\end{aligned}$$

Therefore,

$$log^{[2]}\mu(r, F_n) < (\rho(f_n) + \epsilon) log\mu(2r, F_{n-1}) + O(1).$$

So,

$$log^{[3]}\mu(r, F_n) < (\rho(f_{n-1}) + \epsilon) log\mu(2^2 r, F_{n-2}) + O(1).$$

Therefore,

$$log^{[4]}\mu(r, F_n) < (\rho(f_{n-2}) + \epsilon) log\mu(2^3 r, F_{n-3}) + O(1).$$

After $(n-2)$ steps we get

$$\begin{aligned}
log^{[n]}\mu(r, F_n) &< (\rho(f_2) + \epsilon) log\mu(2^{n-1} r, F_1) + O(1) \\
&= (\rho(f_2) + \epsilon) log\mu(2^{n-1} r, (1-\alpha)z + \alpha f_1) + O(1) \\
&\leq (\rho(f_2) + \epsilon)[log\mu(2^{n-1} r, \alpha f_1) + log\mu(2^{n-1} r, (1-\alpha)z)] + O(1) \\
&= (\rho(f_2) + \epsilon)[log\mu(2^{n-1} r, f_1) + log\mu(2^{n-1} r, z)] + O(1) \\
&= (\rho(f_2) + \epsilon)[log\mu(2^{n-1} r, f_1) + log 2^{n-1} r)] + O(1) \\
&\leq (\rho(f_2) + \epsilon)[log\mu(2^{n-1} r, f_1) + log\mu(2^{n-1} r, f_1)] + O(1) \\
&= 2(\rho(f_2) + \epsilon) log\mu(2^{n-1} r, f_1) + O(1) \\
&< (\rho(f_2) + \epsilon)(2^{n-1} r)^{(\rho(f_1)+\epsilon)} + O(1).
\end{aligned}$$

On the other hand, for a sequence $r = r_n \to \infty$

$$log^{[2]}\mu(r, f_n) < (\rho(f_n) - \epsilon) log r.$$

Expressing $R_n = (log\ r_n)^{\frac{1}{\rho(f_n)}}$ it follows that

$$log^{[2]}\mu(exp(R_n^{\rho(f_n)}), f_n) > (\rho(f_n) - \epsilon) {R_n}^{\rho(f_n)}.$$

Thus for $r = R_n(\geq r_0)$

$$\frac{log^{[n]}\mu(r, F_n)}{log^{[2]}\mu(exp(r^{\rho(f_n)}), f_n)} < \frac{(\rho(f_2)+\epsilon)(2^{n-1}r)^{(\rho(f_1)+\epsilon)} + O(1)}{(\rho(f_n)-\epsilon)r^{\rho(f_n)}}.$$

Hence,

$$\liminf_{r\to\infty} \frac{log^{[n]}\mu(r, F_n)}{log^{[2]}\mu(exp(r^{\rho(f_n)}), f_n)} = 0.$$

Theorem 3.6. *Let $f_1, f_2, ..., f_n$ are entire functions of finite orders with $\lambda(f_1) > \rho(f_k)(1 \leq k \leq n)$ and $\lambda(f_n) > 0$ and suppose $e^{\gamma(\mu(\frac{r}{4}, F_n))^\delta} \geq \mu(r, F_n)$ holds for every $\gamma > 0,\ \delta > 0$ and also for every positive integer $n.$ Then*

$$\lim_{r\to\infty} \frac{log^{[n]}\mu(r, F_n)}{log^{[2]}\mu(exp(r^{\rho(f_k)}), f_k)} = \infty.$$

Proof. We choose ϵ, so that $0 < \epsilon < \lambda(f_1) - \rho(f_k)$. From (3.5) we get for all $r \geq r_0$

$$log^{[2]}\mu(r, F_n) > \frac{1}{2^{n-1}}(\lambda(f_n)-\epsilon)(\lambda(f_{n-1})-\epsilon)...(\lambda(f_2)-\epsilon)[(\frac{r}{4^{n-1}})^{\lambda(f_1)-\epsilon} - log\frac{r}{4^{n-1}}] + O(1).$$

On the other hand for all $r \geq r_0$

$$log^{[2]}\mu(r, f_k) < (\rho(f_k)+\epsilon)\ log\ r$$
$$i.e.,\ log^{[2]}\mu(exp(r^{\rho(f_k)}), f_k) < (\rho(f_k)+\epsilon)\ r^{\rho(f_k)}.$$

Thus for all sufficiently large r

$$\frac{log^{[2]}\mu(r, F_n)}{log^{[2]}\mu(exp(r^{\rho(f_k)}), f_k)} > \frac{\frac{1}{2^{n-1}}(\lambda(f_n)-\epsilon)(\lambda(f_{n-1})-\epsilon)...(\lambda(f_2)-\epsilon)[(\frac{r}{4^{n-1}})^{\lambda(f_1)-\epsilon} - log\frac{r}{4^{n-1}}] + O(1)}{(\rho(f_k)+\epsilon)\ r^{\rho(f_k)}}$$
$$\to \infty\ as\ r \to \infty.$$

Hence,

$$\lim_{r\to\infty} \frac{log^{[2]}\mu(r, F_n)}{log^{[2]}\mu(exp(r^{\rho(f_k)}), f_k)} = \infty.$$

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