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ON RELATIVE ORDER OF ITERATED FUNCTIONS WITH RESPECT TO ITERATED FUNCTIONS OF ENTIRE AND MEROMORPHIC FUNCTIONS

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Abstract. In this paper we prove some results on relative order of iterated functions with respect to iterated functions formed with entire and meromorphic functions.

1. Introduction and Definitions

The Maximum modulus of an entire function $f(z)$ is defined by

$$M_f(r) = max\{|f(z)| : |z| = r\}.$$

If f is non-constant then $M_f(r)$ is strictly increasing and continuous function of r and its inverse

$$M_f^{-1} : (|f(0)|, \infty) \to (0, \infty)$$

exists and is such that

$$\lim_{r\to\infty} M_f(r) = \infty.$$

Definition 1.1. The order of an entire function $f(z)$ is defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r}.$$

In [4] Bernal introduced the definition of relative order of entire functions as follows.

Definition 1.2. [4] If $f(z)$ and $g(z)$ are two entire functions then the relative order of $f(z)$ with respect to $g(z)$ is defined as

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}. \end{aligned}$$

Key words and phrases. Entire function; Meromorphic function; Relative order; Composite function.

If f be meromorphic then Lahiri and Banerjee [7] introduced relative order of f with respect to entire g as follows.

Definition 1.3. [7] The relative order of a meromorphic function $f(z)$ with respect to an entire function $g(z)$ is defined as

$$\begin{aligned} \rho_g(f) &= \inf\{\lambda > 0 : T_f(r) < T_g(r^{\lambda}) \text{ for all } r > r_0(\lambda) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}. \end{aligned}$$

After this, Banerje [1] introduced the definition of relative order of a meromorphic function f with respect to another meromorphic function g as follows.

Definition 1.4. [1] The relative order of a meromorphic function $f(z)$ with respect to a meromorphic function $g(z)$ is defined as

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : T_f(r) < [T_g(r)]^{\mu} \text{ for all r}\} \\ &= \limsup_{r\to\infty} \frac{\log T_f(r)}{\log T_g(r)}. \end{aligned}$$

Here we introduce the following definition.

Definition 1.5. The relative order of an entire function $f(z)$ with respect to a meromorphic function $g(z)$ is defined as

$$\rho_g(f) = \limsup_{r\to\infty} \frac{\log r}{\log T_f^{-1}(T_g(r))}.$$

Very recently [2] Banerjee and Adhikary proved some results on relative order of composite function with respect to composite function formed with entire and meromorphic functions. In the present paper we generalise the results of Banerjee and Adhikary for iterated functions.

2. Known Lemmas

In this section we present some known lemmas which will be needed in the next section.

Lemma 2.1. [6] *Let g be an entire function. Then for all large values of r*

$$T_g(r) \le \log M_g(r) \le 3T_g(2r).$$

Lemma 2.2. [9] *Let f and g be two entire functions. Then for all large values of r*

$$T_{f\circ g}(r) \ge \frac{1}{3} \log M_f(\frac{1}{9} M_g(\frac{r}{4})).$$

Lemma 2.3. [5] *Let f and g be two entire functions. Then for all sufficiently large values of r*

$$M_f(\frac{1}{8}M_g(\frac{r}{2}) - |g(0)|) \le M_{f\circ g}(r) \le M_f(M_g(r)).$$

Lemma 2.4. [3] *If f is meromorphic and g is entire then for all large of values of r*

$$T_{f\circ g}(r) \le \{1 + 0(1)\}\frac{T_g(r)}{\log M_g(r)}T_f(M_g(r)).$$

Lemma 2.5. [8] *Let f and g be two entire functions. If $M_g(r) > \frac{2+\epsilon}{\epsilon}|g(0)|$ for any $\epsilon > 0$, then*

$$T_{f\circ g}(r) < (1+\epsilon)T_f(M_g(r)).$$

In particular if $g(0) = 0$ then

$$T_{f\circ g}(r) < T_f(M_g(r))$$

for all $r > 0$.

3. Preliminary Theorems

In this section we presents some results in the form of preliminary theorems which will be needed in the sequel.

Theorem 3.1. *Let $f_1, f_2, \cdots, f_n$ be entire functions. Then for all large values of r*

$$M_{f_1\circ f_2\circ\cdots\circ f_n}(r) \le M_{f_1}(M_{f_2}(\cdots M_{f_n}(r)\cdots)).$$

i.e;

$$M_{f_n}^{-1}(\cdots M_{f_2}^{-1}(M_{f_1}^{-1}(r))\cdots) \le M_{f_1\circ f_2\circ\cdots\circ f_n}^{-1}(r).$$

Proof. Using Lemma 2.3 successively, we get

$$M_{f_1\circ f_2\circ\cdots\circ f_n}(r) \le M_{f_1}(M_{f_2}(\cdots M_{f_{n-1}}(M_{f_n}(r))\cdots)). \tag{3.1}$$

Taking

$$M_{f_1}(M_{f_2}(\cdots M_{f_{n-1}}(M_{f_n}(r))\cdots)) = R$$

we get

$$r = M_{f_n}^{-1}(M_{f_{n-1}}^{-1}(\cdots M_{f_2}^{-1}(M_{f_1}^{-1}(R))\cdots)).$$

So from (3.1) we get

$$M_{f_n}^{-1}(M_{f_{n-1}}^{-1}(\cdots M_{f_2}^{-1}(M_{f_1}^{-1}(r))\cdots)) \le M_{f_1\circ f_2\circ\cdots\circ f_n}^{-1}(r).$$

Theorem 3.2. *Let $f_1, f_2, \cdots, f_n$ be entire functions. Then for all large values of r*

$$M_{f_1 \circ f_2 \circ \cdots \circ f_n}(r) \geq M_{f_1}(\frac{1}{9} M_{f_2}(\frac{1}{18} M_{f_3} \cdots \frac{1}{18} M_{f_n}(\frac{r}{2}) \cdots)).$$

i.e;

$$2M_{f_n}^{-1}(18M_{f_{n-1}}^{-1}(\cdots 18M_{f_2}^{-1}(9M_{f_1}^{-1}(r))\cdots)) \geq M_{f_1 \circ f_2 \circ \cdots \circ f_n}^{-1}(r).$$

Proof. For two entire functions f_1 and f_2 we get from Lemma 2.3

$$M_{f_1 \circ f_2}(r) \geq M_{f_1}(\frac{1}{9} M_{f_2}(\frac{r}{2})).$$

Now for three entire functions f_1, f_2 and f_3 as in the above we get

$$M_{f_1 \circ f_2 \circ f_3}(r) \geq M_{f_1 \circ f_2}(\frac{1}{9} M_{f_3}(\frac{r}{2})) \geq M_{f_1}(\frac{1}{9} M_{f_2}(\frac{1}{18} M_{f_3}(\frac{r}{2}))).$$

Similarly for n-functions, we get

$$M_{f_1 \circ f_2 \circ \cdots \circ f_n}(r) \geq M_{f_1}(\frac{1}{9} M_{f_2}(\frac{1}{18} M_{f_3}(\cdots \frac{1}{18} M_{f_n}(\frac{r}{2}) \cdots))). \tag{3.2}$$

Taking

$$M_{f_1}(\frac{1}{9} M_{f_2}(\frac{1}{18} M_{f_3}(\cdots \frac{1}{18} M_{f_n}(\frac{r}{2}) \cdots))) = R,$$

we get

$$r = 2M_{f_n}^{-1}(18M_{f_{n-1}}^{-1}(\cdots 18M_{f_2}^{-1}(9M_{f_1}^{-1}(R))\cdots)).$$

So from (3.2) we get

$$2M_{f_n}^{-1}(18M_{f_{n-1}}^{-1}(\cdots 18M_{f_2}^{-1}(9M_{f_1}^{-1}(r))\cdots)) \geq M_{f_1 \circ f_2 \circ \cdots \circ f_n}^{-1}(r).$$

Theorem 3.3. *Let $f_1, f_2, \cdots, f_n$ be entire functions such that $M_{f_i}(r) > \frac{2+\epsilon}{\epsilon}|f_i(0)|$ for $i = 2, 3, ...n$ and for any $\epsilon > 0$. Then for all large values of r*

$$T_{f_1 \circ f_2 \circ \cdots \circ f_n}(r) \leq (1+\epsilon)^{(n-1)} T_{f_1}(M_{f_2}(\cdots M_{f_n}(r) \cdots)).$$

In particular taking $\epsilon = 1$, we get

$$M_{f_n}^{-1}(M_{f_{n-1}}^{-1}(\cdots M_{f_2}^{-1}(T_{f_1}^{-1}(\frac{r}{2^{n-1}}))\cdots)) \leq T_{f_1 \circ f_2 \circ \cdots \circ f_n}^{-1}(r).$$

Proof. For entire functions f_1, f_2, f_3 using Lemma 2.5, we get

$$T_{f_1 \circ f_2}(r) \leq (1+\epsilon) T_{f_1}(M_{f_2}(r)).$$

and

$$T_{f_1 \circ f_2 \circ f_3}(r) \leq (1+\epsilon) T_{f_1 \circ f_2}(M_{f_3}(r)) \leq (1+\epsilon)^2 T_{f_1}(M_{f_2}(M_{f_3}(r))).$$

Similarly for n-functions, we get

$$T_{f_1 \circ f_2 \circ \cdots \circ f_n}(r) \leq (1+\epsilon)^{n-1} T_{f_1}(M_{f_2}(\cdots M_{f_n}(r) \cdots)).$$

In particular taking $\epsilon = 1$, we have

$$T_{f_1 \circ f_2 \circ \cdots \circ f_n}(r) \leq 2^{n-1} T_{f_1}(M_{f_2}(\cdots M_{f_n}(r)\cdots)).$$

Choosing

$$2^{n-1} T_{f_1}(M_{f_2}(\cdots M_{f_n}(r)\cdots)) = R$$

we get

$$r = M_{f_n}^{-1}(M_{f_{n-1}}^{-1}(\cdots M_{f_2}^{-1}(T_{f_1}^{-1}(\frac{R}{2^{n-1}}))\cdots)).$$

Hence

$$M_{f_n}^{-1}(M_{f_n}^{-1}(\cdots M_{f_2}^{-1}(T_{f_1}^{-1}(\frac{r}{2^{n-1}}))\cdots)) \leq T_{f_1 \circ f_2 \circ \cdots \circ f_n}^{-1}(r).$$

Theorem 3.4. *Let $f_1, f_2, \cdots, f_n$ be entire functions. Then for all large values of r*

$$T_{f_1 \circ f_2 \circ \cdots \circ f_n}(r) \geq \frac{1}{3} \log M_{f_1}(\frac{1}{9} M_{f_2}(\cdots \frac{1}{18} M_{f_n}(\frac{r}{8})\cdots)).$$

i.e;

$$8 M_{f_n}^{-1}(18 M_{f_{n-1}}^{-1}(\cdots 9 M_{f_1}^{-1}(\exp 3r)\cdots)) \geq T_{f_1 \circ f_2 \circ \cdots \circ f_n}^{-1}(r).$$

Proof. For entire functions $f_1, f_2, \cdots, f_n$ using Lemma 2.2 we get

$$T_{f_1 \circ f_2 \circ \cdots \circ f_n}(r) \geq \frac{1}{3} \log M_{f_1}(\frac{1}{9} M_{f_2 \circ f_3 \circ \cdots \circ f_n}(\frac{r}{4})).$$

Now using Theorem 3.2 we get

$$T_{f_1 \circ f_2 \circ \cdots \circ f_n}(r) \geq \frac{1}{3} \log M_{f_1}(\frac{1}{9} M_{f_2}(\frac{1}{9} M_{f_3}(\frac{1}{18} M_{f_4}(\cdots \frac{1}{18} M_{f_n}(\frac{r}{8})\cdots)))).$$

i.e;

$$8 M_{f_n}^{-1}(18 M_{f_{n-1}}^{-1}(\cdots 9 M_{f_1}^{-1}(\exp 3r)\cdots)) \geq T_{f_1 \circ f_2 \circ \cdots \circ f_n}^{-1}(r).$$

Theorem 3.5. *Let h_1 be a meromorphic function and $f_2, f_3, \cdots, f_n$ be entire functions. Then for all large values of r*

$$T_{h_1 \circ f_2 \circ \cdots \circ f_n}(r) \leq T_{h_1}(M_{f_2}(M_{f_3}(\cdots M_{f_n}(r)\cdots))).$$

Proof. Since h_1 is meromorphic and $f_2, f_3, \cdots, f_n$ are entire functions, we have from Lemma 2.4

$$T_{h_1 \circ f_2 \circ \cdots \circ f_n}(r) \leq T_{h_1}(M_{f_2 \circ f_3 \circ \cdots \circ f_n}(r)).$$

Now from Theorem 3.1, we get

$$T_{h_1 \circ f_2 \circ \cdots \circ f_n}(r) \leq T_{h_1}(M_{f_2}(M_{f_3}(\cdots M_{f_n}(r)\cdots))).$$

4. Main Theorems

In this section we present main results of the paper.

Theorem 4.1. *Let $f_1, f_2, \cdots, f_n$ and $h_1, h_2, \cdots, h_n$ be entire functions of respective finite nonzero orders and g be a polynomial of degree m. Then the relative order of $h_1 \circ h_2 \circ \cdots \circ h_n$ with respect to $f_1 \circ f_2 \circ \cdots \circ f_n \circ g$ satisfies the inequality*

$$\frac{\lambda_{h_n}}{m\rho_{f_n}} \leq \rho_{f_1 \circ f_2 \circ \cdots \circ f_n \circ g}(h_1 \circ h_2 \circ \cdots \circ h_n) \leq \frac{\rho_{h_n}}{m\lambda_{f_n}}.$$

Proof. We have by definition of order for any $\epsilon > 0$ there exists $r_0(\epsilon) > 0$ such that

$$M_{f_1}(r) < \exp\{r^{\rho_{f_1}+\epsilon}\} \quad \text{for all} \quad r > r_0(\epsilon) \tag{4.1}$$

$$i.e; \quad M_{f_1}^{-1}(r) > \exp\{\frac{1}{\rho_{f_1}+\epsilon} \log^{[2]} r\}. \tag{4.2}$$

Similarly

$$M_{f_2}^{-1}(r) > \exp\{\frac{1}{\rho_{f_2}+\epsilon} \log^{[2]} r\}; \tag{4.3}$$

$$\vdots$$

$$M_{f_n}^{-1}(r) > \exp\{\frac{1}{\rho_{f_n}+\epsilon} \log^{[2]} r\}. \tag{4.4}$$

Again for arbitrary $\epsilon > 0$ and for all large values of r

$$M_{h_1}(r) > \exp\{r^{\lambda_{h_1}-\epsilon}\}; \tag{4.5}$$

$$M_{h_2}(r) > \exp\{r^{\lambda_{h_2}-\epsilon}\}; \tag{4.6}$$

$$\vdots$$

$$M_{h_n}(r) > \exp\{r^{\lambda_{h_n}-\epsilon}\}. \tag{4.7}$$

Let $g(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$.
Then for any $\epsilon > 0$ there exists $r_1(\epsilon) > 0$ such that

$$|a_m| r^m (1-\epsilon) < M_g(r) < |a_m| r^m (1+\epsilon) \text{ for all } r > r_1(\epsilon). \tag{4.8}$$

So

$$M_g^{-1}(r) > \{\frac{r}{|a_m|(1+\epsilon)}\}^{\frac{1}{m}} \quad \text{and} \quad M_g^{-1}(r) < \{\frac{r}{|a_m|(1-\epsilon)}\}^{\frac{1}{m}}. \tag{4.9}$$

So

$$\rho_{f_1\circ f_2\circ\cdots\circ f_n\circ g}(h_1\circ h_2\circ\cdots\circ h_n)=\limsup_{r\to\infty}\frac{\log M^{-1}_{f_1\circ f_2\circ\cdots\circ f_n\circ g}(M_{h_1\circ h_2\circ\cdots\circ h_n}(r))}{\log r}$$

$$\geq limsup_{r\to\infty}\frac{\log M_g^{-1}(M_{f_n}^{-1}(\cdots M_{f_2}^{-1}(M_{f_1}^{-1}(M_{h_1\circ h_2\circ\cdots\circ h_n}(r)))\cdots))}{\log r}\ \text{(from Theorem 3.1)}$$

$$\geq limsup_{r\to\infty}\frac{\log M_g^{-1}(M_{f_n}^{-1}(\cdots M_{f_2}^{-1}(M_{f_1}^{-1}(M_{h_1}(\frac{1}{9}M_{h_2}(\cdots\frac{1}{18}M_{h_n}(\frac{r}{2})\cdots))))\cdots))}{\log r}\text{(from Theorem 3.2)}$$

$$\geq limsup_{r\to\infty}\frac{\log M_g^{-1}(M_{f_n}^{-1}(\cdots M_{f_2}^{-1}(M_{f_1}^{-1}(M_{h_1}(\frac{1}{9}M_{h_2}(\cdots(\frac{1}{18}\exp(\frac{r}{2})^{\lambda_{h_n}-\epsilon})\cdots))))\cdots))}{\log r}\quad\text{(from 4.7)}$$

$$\geq limsup_{r\to\infty}\frac{\log M_g^{-1}(M_{f_n}^{-1}(\cdots M_{f_2}^{-1}(M_{f_1}^{-1}(M_{h_1}(\frac{1}{9}M_{h_2}(\cdots(\exp[\frac{1}{18}\exp(\frac{r}{2})^{\lambda_{h_n}-\epsilon}]^{\lambda_{h_{n-1}}-\epsilon})\cdots))))\cdots))}{\log r}$$

$$\geq limsup_{r\to\infty}\frac{\begin{array}{l}\log M_g^{-1}(M_{f_n}^{-1}(\cdots(M_{f_2}^{-1}(M_{f_1}^{-1}(\exp[\frac{1}{9}\exp[\frac{1}{18}\exp(\cdots\\ \quad\cdots[\frac{1}{18}\exp(\frac{r}{2})^{\lambda_{h_n}-\epsilon}]^{\lambda_{h_{n-1}}-\epsilon}\cdots)]^{\lambda_{h_1}-\epsilon}])))\cdots))\end{array}}{\log r}\quad\text{(from 4.5)}$$

$$\geq limsup_{r\to\infty}\frac{\begin{array}{l}\log M_g^{-1}(M_{f_n}^{-1}(\cdots(M_{f_2}^{-1}(\exp\{\frac{1}{\rho_{f_1}+\epsilon}\log^{[2]}(\exp[\frac{1}{9}\exp[\frac{1}{18}\exp(\cdots\\ \quad\cdots[\frac{1}{18}\exp(\frac{r}{2})^{\lambda_{h_n}-\epsilon}]^{\lambda_{h_{n-1}}-\epsilon}\cdots)]^{\lambda_{h_1}-\epsilon}])\})\cdots))\end{array}}{\log r}\quad\text{(from 4.2)}$$

$$\geq limsup_{r\to\infty}\frac{\log M_g^{-1}(M_{f_n}^{-1}(\cdots M_{f_2}^{-1}(\exp\{\frac{\lambda_{h_1}-\epsilon}{\rho_{f_1}+\epsilon}(\frac{1}{9}\exp[\frac{1}{18}\exp\cdots[\frac{1}{18}\exp(\frac{r}{2})^{\lambda_{h_n}-\epsilon}]^{\lambda_{h_{n-1}}-\epsilon}\cdots]^{\lambda_{h_2}-\epsilon})\})\cdots))}{\log r}$$

$$\geq limsup_{r\to\infty}\frac{\begin{array}{l}\log M_g^{-1}(M_{f_n}^{-1}(\cdots(M_{f_2}^{-1}(\exp\{\frac{1}{\rho_{f_2}+\epsilon}\log^{[2]}(\exp\{\frac{\lambda_{h_1}-\epsilon}{\rho_{f_1}+\epsilon}\exp[\frac{1}{9}\exp[\frac{1}{18}\exp\\ \quad\cdots[\frac{1}{18}\exp(\frac{r}{2})^{\lambda_{h_n}-\epsilon}]^{\lambda_{h_{n-1}}-\epsilon}\cdots]^{\lambda_{h_2}-\epsilon}]\}\})\cdots))\end{array}}{\log r}$$

$$\geq limsup_{r\to\infty}\frac{\log M_g^{-1}(M_{f_n}^{-1}(\cdots M_{f_3}^{-1}(\exp\{\frac{\lambda_{h_2}-\epsilon}{\rho_{f_2}+\epsilon}(\frac{1}{9}\exp[\frac{1}{18}\exp(\cdots[\frac{1}{18}\exp(\frac{r}{2})^{\lambda_{h_n}-\epsilon}]^{\lambda_{h_{n-1}}-\epsilon}\cdots)]^{\lambda_{h_3}-\epsilon})\})\cdots))}{\log r}$$

$$\vdots$$

$$\geq limsup_{r\to\infty}\frac{\log M_g^{-1}(M_{f_n}^{-1}(\exp\{\frac{1}{\rho_{f_{n-1}}+\epsilon}\log^{[2]}(\exp(\frac{1}{9}\exp(\frac{r}{2})^{\lambda_{h_n}-\epsilon})^{\lambda_{h_{n-1}}-\epsilon})\}))+O(1)}{\log r}$$

$$\geq limsup_{r\to\infty}\frac{\log M_g^{-1}(M_{f_n}^{-1}(\exp\{\frac{\lambda_{h_{n-1}}-\epsilon}{\rho_{f_{n-1}}+\epsilon}(\frac{r}{2})^{\lambda_{h_n}-\epsilon}\}))+O(1)}{\log r}$$

$$\geq limsup_{r\to\infty}\frac{\log M_g^{-1}\{\frac{1}{\rho_{f_n}+\epsilon}\log^{[2]}(\exp\{\frac{\lambda_{h_{n-1}}-\epsilon}{\rho_{f_{n-1}}+\epsilon}(\frac{r}{2})^{\lambda_{h_n}-\epsilon}\})\}+O(1)}{\log r}$$

$$\geq limsup_{r\to\infty}\frac{\log M_g^{-1}(\exp\{\frac{\lambda_{h_n}-\epsilon}{\rho_{f_n}+\epsilon}\log(\frac{r}{2})\})+O(1)}{\log r}$$

$$\geq limsup_{r\to\infty}\frac{\log M_g^{-1}\{(\frac{r}{2})^{\frac{\lambda_{h_n}-\epsilon}{\rho_{f_n}+\epsilon}}\}+O(1)}{\log r}$$

$$\geq limsup_{r\to\infty}\frac{\log\{\frac{(\frac{r}{2})^{\frac{\lambda_{h_n}-\epsilon}{\rho_{f_n}+\epsilon}}}{|a_m|(1+\epsilon)}\}^{\frac{1}{m}}+O(1)}{\log r}\text{(from 4.9)}$$

$$\geq limsup_{r\to\infty}\frac{1}{m}\frac{\log\{(\frac{r}{2})^{\frac{\lambda_{h_n}-\epsilon}{\rho_{f_n}+\epsilon}}\}-\log|a_m|(1+\epsilon)+O(1)}{\log r}$$

$$\geq limsup_{r\to\infty}\frac{1}{m}\frac{\frac{\lambda_{h_n}-\epsilon}{\rho_{f_n}+\epsilon}\log(\frac{r}{2})-\log|a_m|(1+\epsilon)+O(1)}{\log r}$$

$$\geq\frac{1}{m}\frac{\lambda_{h_n}-\epsilon}{\rho_{f_n}+\epsilon}$$

$$\geq\frac{1}{m}\frac{\lambda_{h_n}}{\rho_{f_n}}\quad\text{since }\epsilon>0\text{ is arbitrary.}$$

Again for all sufficiently large values of r

$$M_{h_1}(r) \leq \exp\{r^{\rho_{h_1}+\epsilon}\},\ M_{h_2}(r) \leq \exp\{r^{\rho_{h_2}+\epsilon}\},\ \cdots\ , M_{h_n}(r) \leq \exp\{r^{\rho_{h_n}+\epsilon}\} \tag{4.10}$$

and

(4.11)

$$M_{f_1}^{-1}(r) < \exp\{\frac{1}{\lambda_{f_1}-\epsilon}\log^{[2]} r\}, M_{f_2}^{-1}(r) < \exp\{\frac{1}{\lambda_{f_2}-\epsilon}\log^{[2]} r\}, \cdots , M_{f_2}^{-1}(r) < \exp\{\frac{1}{\lambda_{f_2}-\epsilon}\log^{[2]} r\}$$

Now

$$\rho_{f_1\circ f_2\circ\cdots\circ f_n\circ g}(h_1\circ h_2\circ\cdots\circ h_n) = \limsup_{r\to\infty}\frac{\log M^{-1}_{f_1\circ f_2\circ\cdots\circ f_n\circ g}(M_{h_1\circ h_2\circ\cdots\circ h_n}(r))}{\log r}$$

$$\leq limsup_{r\to\infty}\frac{\log(2M_g^{-1}(18M_{f_n}^{-1}(18M_{f_{n-1}}^{-1}(\cdots 18M_{f_2}^{-1}(9M_{f_1}^{-1}(M_{h_1\circ h_2\circ\cdots\circ h_n}(r)))\cdots))))}{\log r}(\text{ from Theorem } 3.2)$$

$$\leq limsup_{r\to\infty}\frac{\begin{array}{l}\log(2M_g^{-1}(18M_{f_n}^{-1}(18M_{f_{n-1}}^{-1}(\cdots\\ \quad\cdots 18M_{f_2}^{-1}(9M_{f_1}^{-1}(M_{h_1}(M_{h_2}(\cdots M_{h_n}(r)\cdots))))\cdots))))\end{array}}{\log r}(\text{ from Theorem } 3.1)$$

$$\leq limsup_{r\to\infty}\frac{\begin{array}{l}\log(2M_g^{-1}(18M_{f_n}^{-1}(18M_{f_{n-1}}^{-1}(\cdots\\ \quad\cdots 18M_{f_2}^{-1}(9M_{f_1}^{-1}(M_{h_1}(M_{h_2}(\cdots M_{h_{n-1}}(\exp(r^{\rho_{h_n}+\epsilon}))\cdots))))\cdots)))) + O(1)\end{array}}{\log r}(\text{from } 4.10)$$

$$\leq limsup_{r\to\infty}\frac{\begin{array}{l}\log(2M_g^{-1}(18M_{f_n}^{-1}(18M_{f_{n-1}}^{-1}(\cdots\\ \quad\cdots 18M_{f_2}^{-1}(9M_{f_1}^{-1}(M_{h_1}(M_{h_2}(\cdots M_{h_{n-2}}(\exp[\exp(r^{\rho_{h_n}+\epsilon})]^{\rho_{h_{n-1}}+\epsilon})\cdots))))\cdots)))) + O(1)\end{array}}{\log r}$$

$$\leq limsup_{r\to\infty}\frac{\begin{array}{l}\log(2M_g^{-1}(18M_{f_n}^{-1}(18M_{f_{n-1}}^{-1}(\cdots\\ \quad\cdots 18M_{f_2}^{-1}(9M_{f_1}^{-1}[\exp(\exp(\cdots(\exp(r^{\rho_{h_n}+\epsilon}))\cdots))^{\rho_{h_1}+\epsilon}])\cdots)))) + O(1)\end{array}}{\log r}$$

$$\leq limsup_{r\to\infty}\frac{\begin{array}{l}\log(2M_g^{-1}(18M_{f_n}^{-1}(18M_{f_{n-1}}^{-1}(\cdots\\ \quad\cdots(18\exp\{\frac{1}{\lambda_{f_1}-\epsilon}\log^{[2]}[\exp(\exp(\cdots(\exp(r^{\rho_{h_n}+\epsilon}))\cdots))^{\rho_{h_1}+\epsilon}]\})\cdots)))) + O(1)\end{array}}{\log r}(\text{from } 4.11)$$

$$\leq limsup_{r\to\infty}\frac{\begin{array}{l}\log(2M_g^{-1}(18M_{f_n}^{-1}(18M_{f_{n-1}}^{-1}(\cdots\\ \quad\cdots(18\exp\{\frac{\rho_{h_1}+\epsilon}{\lambda_{f_1}-\epsilon}[\exp(\exp(\cdots(\exp(r^{\rho_{h_n}+\epsilon}))\cdots))^{\rho_{h_2}+\epsilon}]\})\cdots)))) + O(1)\end{array}}{\log r}$$

$$\vdots$$

$$\leq limsup_{r\to\infty}\frac{\log(2M_g^{-1}(18M_{f_n}^{-1}(18\exp\{\frac{1}{\lambda_{f_{n-1}}-\epsilon}\log^{[2]}(\exp(\exp(r^{\rho_{h_n}+\epsilon})))^{\rho_{h_{n-1}}+\epsilon}\}))) + O(1)}{\log r}$$

$$\leq limsup_{r\to\infty}\frac{\log(2M_g^{-1}(18M_{f_n}^{-1}(18\exp\{\frac{\rho_{h_{n-1}}+\epsilon}{\lambda_{f_{n-1}}-\epsilon}(r^{\rho_{h_n}+\epsilon})\}))) + O(1)}{\log r}$$

$$\leq limsup_{r\to\infty}\frac{\log(2M_g^{-1}(18r^{\frac{\rho_{h_n}+\epsilon}{\lambda_{f_n}-\epsilon}})) + O(1)}{\log r}$$

$$\leq limsup_{r\to\infty}\frac{\log\{\frac{18r^{\frac{\rho_{h_n}+\epsilon}{\lambda_{f_n}-\epsilon}}}{|a_m|(1-\epsilon)}\}^{\frac{1}{m}} + O(1)}{\log r}(\text{from } 4.9)$$

$$\leq limsup_{r\to\infty}\frac{1}{m}\frac{\log r^{\frac{\rho_{h_n}+\epsilon}{\lambda_{f_n}-\epsilon}}-\log|a_m|(1-\epsilon)+O(1)}{\log r}$$
$$\leq limsup_{r\to\infty}\frac{1}{m}\frac{\frac{\rho_{h_n}+\epsilon}{\lambda_{f_n}-\epsilon}\log r+O(1)}{\log r}$$
$$\leq \frac{1}{m}\frac{\rho_{h_n}+\epsilon}{\lambda_{f_n}-\epsilon}$$
$$\leq \frac{1}{m}\frac{\rho_{h_n}}{\lambda_{f_n}} \text{ since } \epsilon>0 \text{ is arbitrary.}$$

Theorem 4.2. *Let $f_1, f_2, \cdots, f_n, h_1, h_2, \cdots, h_n$ be entire functions of respective finite nonzero orders and g_1, g_2 be two polynomials of degree m_1, m_2 respectively. Then the relative order of $h_1 \circ h_2 \circ \cdots \circ h_n \circ g_2$ with respect to $f_1 \circ f_2 \circ \cdots \circ f_n \circ g_1$ satisfies the inequality*

$$\frac{m_2}{m_1}\frac{\lambda_{h_n}}{\rho_{f_n}} \leq \rho_{f_1\circ f_2\circ\cdots\circ f_n\circ g_1}(h_1\circ h_2\circ\cdots\circ h_n\circ g_2) \leq \frac{m_2}{m_1}\frac{\rho_{h_n}}{\lambda_{f_n}}.$$

Proof: Let

$$g_1(z)=a_0+a_1z+a_2z^2+\cdots+a_{m_1}z^{m_1}$$

and

$$g_2(z)=b_0+b_1z+b_2z^2+\cdots+b_{m_2}z^{m_2}$$

be two polynomials of degree m_1, m_2 respectively. Take $R=|b_{m_2}|(1-\epsilon)(\frac{r}{2})^{m_2}$
By the definition of relative order of an entire function with respect to another entire function we have

$$\rho_{f_1\circ f_2\circ\cdots\circ f_n\circ g_1}(h_1\circ h_2\circ\cdots\circ h_n\circ g_2)=\limsup_{r\to\infty}\frac{\log M^{-1}_{f_1\circ f_2\circ\cdots\circ f_n\circ g_1}(M_{h_1\circ h_2\circ\cdots\circ h_n\circ g_2}(r))}{\log r}$$
$$\geq limsup_{r\to\infty}\frac{\log M^{-1}_{g_1}(M^{-1}_{f_n}(\cdots M^{-1}_{f_2}(M^{-1}_{f_1}(M_{h_1\circ h_2\circ\cdots\circ h_n\circ g_2}(r)))\cdots))}{\log r}(\text{ from Theorem 3.1})$$
$$\geq limsup_{r\to\infty}\frac{\log M^{-1}_{g_1}(M^{-1}_{f_n}(\cdots M^{-1}_{f_2}(M^{-1}_{f_1}(M_{h_1}(\frac{1}{9}M_{h_2}(\cdots\frac{1}{18}M_{h_n}(\frac{1}{18}M_{g_2}(\frac{r}{2}))\cdots))))\cdots))}{\log r}(\text{ from Theorem 3.2})$$
$$\geq limsup_{r\to\infty}\frac{\log M^{-1}_{g_1}(M^{-1}_{f_n}(\cdots M^{-1}_{f_2}(M^{-1}_{f_1}(M_{h_1}(\frac{1}{9}M_{h_2}(\cdots\frac{1}{18}M_{h_n}(|b_{m_2}|(1-\epsilon)(\frac{r}{2})^{m_2})\cdots))))\cdots))+O(1)}{\log r}$$
$$\geq limsup_{r\to\infty}\frac{\log M^{-1}_{g_1}(M^{-1}_{f_n}(\cdots M^{-1}_{f_2}(M^{-1}_{f_1}(M_{h_1}(\frac{1}{9}M_{h_2}(\cdots \frac{1}{18}M_{h_{n-1}}(\frac{1}{18}\exp R^{\lambda_{h_n}-\epsilon})\cdots))))\cdots))+O(1)}{\log r}(\text{from 4.7})$$
$$\geq limsup_{r\to\infty}\frac{\log M^{-1}_{g_1}(M^{-1}_{f_n}(\cdots M^{-1}_{f_2}(M^{-1}_{f_1}(M_{h_1}(\frac{1}{9}M_{h_2}(\cdots \cdots\frac{1}{18}M_{h_{n-2}}(\frac{1}{18}\exp\{\exp(R^{\lambda_{h_n}-\epsilon})\}^{\lambda_{h_{n-1}}-\epsilon})\cdots))))\cdots))+O(1)}{\log r}$$

$$\geq limsup_{r\to\infty}\frac{\log M_{g_1}^{-1}(M_{f_n}^{-1}(\cdots M_{f_2}^{-1}(M_{f_1}^{-1}(\exp(\frac{1}{9}\exp(\cdots \cdots(\exp\{\exp(R^{\lambda_{h_n}-\epsilon})\}^{\lambda_{h_{n-1}}-\epsilon})\cdots)^{\lambda_{h_2}-\epsilon})^{\lambda_{h_1}-\epsilon}))\cdots))+O(1)}{\log r}$$

$$\geq limsup_{r\to\infty}\frac{\log M_{g_1}^{-1}(M_{f_n}^{-1}(\cdots M_{f_2}^{-1}(\exp\{\frac{1}{\rho_{f_1}+\epsilon}\log^{[2]}(\exp(\cdots \cdots(\exp\{\exp(R^{\lambda_{h_n}-\epsilon})\}^{\lambda_{h_{n-1}}-\epsilon})\cdots)^{\lambda_{h_1}-\epsilon})\})\cdots))+O(1)}{\log r}$$

$$\geq limsup_{r\to\infty}\frac{\log M_{g_1}^{-1}(M_{f_n}^{-1}(\cdots M_{f_2}^{-1}(\exp\{\frac{\lambda_{h_1}-\epsilon}{\rho_{f_1}+\epsilon}\log(\frac{1}{9}\exp(\cdots \cdots(\exp\{\exp(R^{\lambda_{h_n}-\epsilon})\}^{\lambda_{h_{n-1}}-\epsilon})\cdots)^{\lambda_{h_2}-\epsilon})\})\cdots))+O(1)}{\log r}$$

$$\geq limsup_{r\to\infty}\frac{\log M_{g_1}^{-1}(M_{f_n}^{-1}(\cdots M_{f_3}^{-1}(\exp\{\frac{1}{\rho_{f_2}+\epsilon}\log^{[2]}[\exp\{\frac{\lambda_{h_1}-\epsilon}{\rho_{f_1}+\epsilon}\exp(\cdots \cdots(\exp\{\exp(R^{\lambda_{h_n}-\epsilon})\}^{\lambda_{h_{n-1}}-\epsilon})\cdots)^{\lambda_{h_2}-\epsilon}\}]\})\cdots))+O(1)}{\log r}$$

$$\vdots$$

$$\geq limsup_{r\to\infty}\frac{\log M_{g_1}^{-1}(M_{f_n}^{-1}(\exp\{\frac{1}{\rho_{f_{n-1}}+\epsilon}\log^{[2]}[\exp(R^{\lambda_{h_n}-\epsilon})]^{\lambda_{h_{n-1}}-\epsilon}\}))+O(1)}{\log r}$$

$$\geq limsup_{r\to\infty}\frac{\log M_{g_1}^{-1}(M_{f_n}^{-1}(\exp\{\frac{\lambda_{h_{n-1}}-\epsilon}{\rho_{f_{n-1}}+\epsilon}\log^{[2]}[\exp(R^{\lambda_{h_n}-\epsilon})]\}))+O(1)}{\log r}$$

$$\geq limsup_{r\to\infty}\frac{\log M_{g_1}^{-1}(\exp\{\frac{1}{\rho_{f_n}+\epsilon}\log^{[2]}[\exp\{\frac{\lambda_{h_{n-1}}-\epsilon}{\rho_{f_{n-1}}+\epsilon}R^{\lambda_{h_n}-\epsilon}\}]\})+O(1)}{\log r}$$

$$\geq limsup_{r\to\infty}\frac{\log M_{g_1}^{-1}(\exp\{\frac{\lambda_{h_n}-\epsilon}{\rho_{f_n}+\epsilon}\log(\frac{r}{2})^{m_2}\})+O(1)}{\log r}$$

$$\geq limsup_{r\to\infty}\frac{\log M_{g_1}^{-1}((\frac{r}{2})^{m_2\frac{\lambda_{h_n}-\epsilon}{\rho_{f_n}+\epsilon}})+O(1)}{\log r}(\text{from } 4.9)$$

$$\geq limsup_{r\to\infty}\frac{\log\{\frac{(\frac{r}{2})^{m_2\frac{\lambda_{h_n}-\epsilon}{\rho_{f_n}+\epsilon}}}{|a_{m_1}|(1+\epsilon)}\}^{\frac{1}{m_1}}+O(1)}{\log r}$$

$$\geq limsup_{r\to\infty}\frac{\frac{m_2}{m_1}\frac{\lambda_{h_n}-\epsilon}{\rho_{f_n}+\epsilon}\log r-\log|a_{m_1}|(1+\epsilon)+O(1)}{\log r}$$

$$\geq\frac{m_2}{m_1}\frac{\lambda_{h_n}}{\rho_{f_n}}\quad\text{since } \epsilon>0 \text{ is any arbitrary.}$$

Using the same arguments as in Theorem 4.1 we can show that
$$\rho_{f_1\circ f_2\circ\cdots\circ f_n\circ g_1}(h_1\circ h_2\circ\cdots\circ h_n\circ g_2)\le\frac{m_2}{m_1}\frac{\rho_{h_n}}{\lambda_{f_n}}.$$

Theorem 4.3. *Let $f_1,f_2,\cdots,f_n$, $h_2,h_3,\cdots,h_n$ be entire functions and g_1 be meromorphic function of respective finite nonzero orders and g_2 be a polynomial of degree m. Then the relative order of $g_1\circ h_2\circ\cdots\circ h_n$ with respect to $f_1\circ f_2\circ\cdots\circ f_n\circ g_2$ satisfies the inequality*
$$\rho_{f_1\circ f_2\circ\cdots\circ f_n\circ g_2}(g_1\circ h_2\circ\cdots\circ h_n)\le\frac{\rho_{h_n}}{m\lambda_{f_n}}.$$

Proof: We know from the definition of lower order, for all large values of r
$$M_{f_1}^{-1}(r)<\exp\{\frac{1}{\lambda_{f_1}-\epsilon}\log^{[2]}r\}.\tag{4.12}$$
Similarly for all large values of r
$$M_{f_2}^{-1}(r)<\exp\{\frac{1}{\lambda_{f_2}-\epsilon}\log^{[2]}r\}.\tag{4.13}$$
$$\vdots$$
$$M_{f_n}^{-1}(r)<\exp\{\frac{1}{\lambda_{f_n}-\epsilon}\log^{[2]}r\}.\tag{4.14}$$
Also
$$T_{g_1}(r)<r^{\rho_{g_1}+\epsilon},\ \text{for all large values of r.}\tag{4.15}$$
Let $g_2(z)=a_0+a_1z+a_2z^2+\cdots+a_mz^m$.
Now
$$\rho_{f_1\circ f_2\circ\cdots\circ f_n\circ g_2}(g_1\circ h_2\circ\cdots\circ h_n)=\limsup_{r\to\infty}\frac{\log T^{-1}_{f_1\circ f_2\circ\cdots\circ f_n\circ g_2}(T_{g_1\circ h_2\circ\cdots\circ h_n}(r))}{\log r}$$
$$\le limsup_{r\to\infty}\frac{\log(8M_{g_2}^{-1}(18M_{f_n}^{-1}(\cdots 18M_{f_2}^{-1}(9M_{f_1}^{-1}(\exp(3T_{g_1\circ h_2\circ\cdots\circ h_n}(r))))\cdots)))}{\log r}\text{(from Theorem 3.4)}$$
$$\le limsup_{r\to\infty}\frac{\begin{array}{l}\log(8M_{g_2}^{-1}(18M_{f_n}^{-1}(\cdots\\ \quad\cdots 18M_{f_2}^{-1}(9M_{f_1}^{-1}(\exp[3T_{g_1}(M_{h_2}(M_{h_3}(\cdots M_{h_n}(r)\cdots)))]))\cdots)))\end{array}}{\log r}\text{(from Theorem 3.5)}$$
$$\le limsup_{r\to\infty}\frac{\begin{array}{l}\log(8M_{g_2}^{-1}(18M_{f_n}^{-1}(\cdots\\ \quad\cdots 18M_{f_2}^{-1}(9M_{f_1}^{-1}(\exp[3T_{g_1}(M_{h_2}(M_{h_3}(\cdots M_{h_{n-1}}(\exp(r^{\rho_{h_n}+\epsilon}))\cdots)))]))\cdots)))\end{array}}{\log r}\text{(from 4.1)}$$
$$\le limsup_{r\to\infty}\frac{\log(8M_{g_2}^{-1}(18M_{f_n}^{-1}(\cdots 18M_{f_2}^{-1}(9M_{f_1}^{-1}(\exp[3T_{g_1}(\exp(\exp(\cdots(\exp(r^{\rho_{h_n}+\epsilon}))\cdots)))^{\rho_{h_2}+\epsilon}]))\cdots))}{\log r}$$
$$\le limsup_{r\to\infty}\frac{\begin{array}{l}\log(8M_{g_2}^{-1}(18M_{f_n}^{-1}(\cdots\\ \quad\cdots 18M_{f_2}^{-1}(9M_{f_1}^{-1}(\exp[\exp(\exp(\exp(\cdots(\exp(r^{\rho_{h_n}+\epsilon}))\cdots)))^{\rho_{h_2}+\epsilon}]^{\rho_{g_1}+\epsilon}))\cdots)))\end{array}}{\log r}$$
$$\le limsup_{r\to\infty}\frac{\log(8M_{g_2}^{-1}(18M_{f_n}^{-1}(\cdots 18M_{f_2}^{-1}(\exp\{\frac{1}{\lambda_{f_1}-\epsilon}\log^{[2]}(\exp[\exp(\cdots\exp(r^{\rho_{h_n}+\epsilon})\cdots]^{\rho_{g_1}+\epsilon})\})\cdots)))+O(1)}{\log r}$$

$$\leq limsup_{r\to\infty}\frac{\log(8M_{g_2}^{-1}(18M_{f_n}^{-1}(\cdots 18M_{f_2}^{-1}(\exp\{\frac{\rho_{g_1}+\epsilon}{\lambda_{f_1}-\epsilon}(\exp[\exp(\cdots\exp(r^{\rho_{h_n}+\epsilon})\cdots)^{\rho_{h_2}+\epsilon}])\})\cdots)))+O(1)}{\log r}$$

$$\leq limsup_{r\to\infty}\frac{\begin{array}{l}\log(8M_{g_2}^{-1}(18M_{f_n}^{-1}(\cdots\\ \cdots 18M_{f_3}^{-1}(\exp\{\frac{1}{\lambda_{f_2}-\epsilon}\log^{[2]}\{\frac{\rho_{g_1}+\epsilon}{\lambda_{f_1}-\epsilon}(\exp[\exp(\cdots\exp(r^{\rho_{h_n}+\epsilon})\cdots)^{\rho_{h_2}+\epsilon}])\}\})\cdots)))+O(1)\end{array}}{\log r}$$

$$\leq limsup_{r\to\infty}\frac{\log(8M_{g_2}^{-1}(18M_{f_n}^{-1}(\cdots 18M_{f_3}^{-1}(\exp\{\frac{\rho_{h_2}+\epsilon}{\lambda_{f_2}-\epsilon}(\exp[\exp(\cdots\exp(r^{\rho_{h_n}+\epsilon})\cdots)^{\rho_{h_3}+\epsilon}])\})\cdots)))+O(1)}{\log r}$$

$$\vdots$$

$$\leq limsup_{r\to\infty}\frac{\log(8M_{g_2}^{-1}(18M_{f_n}^{-1}(\exp\{\frac{\rho_{h_{n-1}}+\epsilon}{\lambda_{f_{n-1}}-\epsilon}r^{\rho_{f_n}+\epsilon}\})))+O(1)}{\log r}$$

$$\leq limsup_{r\to\infty}\frac{\log(8M_{g_2}^{-1}(\exp\{\frac{1}{\lambda_{f_n}-\epsilon}\log^{[2]}(\exp\{\frac{\rho_{h_{n-1}}+\epsilon}{\lambda_{f_{n-1}}-\epsilon}r^{\rho_{f_n}+\epsilon}\})\}))+O(1)}{\log r}$$

$$\leq limsup_{r\to\infty}\frac{\log(8M_{g_2}^{-1}(\exp\{\frac{1}{\lambda_{f_n}-\epsilon}\log\{\frac{\rho_{h_{n-1}}+\epsilon}{\lambda_{f_{n-1}}-\epsilon}r^{\rho_{f_n}+\epsilon}\}\}))+O(1)}{\log r}$$

$$\leq limsup_{r\to\infty}\frac{\log(8M_{g_2}^{-1}(\exp\{\frac{\rho_{f_n}+\epsilon}{\lambda_{f_n}-\epsilon}\log r\}))+O(1)}{\log r}$$

$$\leq limsup_{r\to\infty}\frac{\log(8M_{g_2}^{-1}(r^{\frac{\rho_{f_n}+\epsilon}{\lambda_{f_n}-\epsilon}}))+O(1)}{\log r}$$

$$\leq limsup_{r\to\infty}\frac{\frac{1}{m}\log(\frac{r^{\frac{\rho_{f_n}+\epsilon}{\lambda_{f_n}-\epsilon}}}{|a_m|(1-\epsilon_1)})+O(1)}{\log r}(\text{from } 4.9)$$

$$\leq limsup_{r\to\infty}\frac{\frac{1}{m}(\log r^{\frac{\rho_{f_n}+\epsilon}{\lambda_{f_n}-\epsilon}}-\log|a_m|(1-\epsilon_1))+O(1)}{\log r}$$

$$\leq limsup_{r\to\infty}\frac{\frac{1}{m}\frac{\rho_{f_n}+\epsilon}{\lambda_{f_n}-\epsilon}\log r+O(1)}{\log r}$$

$$\leq\frac{1}{m}\frac{\rho_{f_n}+\epsilon}{\lambda_{f_n}-\epsilon}$$

$$\leq\frac{1}{m}\frac{\rho_{h_n}}{\lambda_{f_n}},\quad \text{as } \epsilon>0 \text{ is arbitrary.}$$

Theorem 4.4. *Let $f_1, f_2, \cdots, f_n$, $h_2, h_3, \cdots, h_n$ be entire functions and g_1 be meromorphic function of respective finite non-zero orders and g_2 be a polynomial of degree m. Then the relative order of $f_1 \circ f_2 \circ \cdots \circ f_n \circ g_2$ with respect to $g_1 \circ h_2 \circ \cdots \circ h_n$ satisfies the inequality*

$$\rho_{g_1\circ h_2\circ\cdots\circ h_n}(f_1\circ f_2\circ\cdots\circ f_n\circ g_2)\geq m\frac{\lambda_{f_n}}{\rho_{h_n}}.$$

Proof: Let $g_2(z) = a_0 + a_1z + a_2z^2 + \cdots + a_mz^m$ be a polynomial of degree m.
By the definition of relative order of an entire function with respect to a meromorphic

function we get

$$\rho_{f_1\circ f_2\circ\cdots\circ f_n\circ g_2}(g_1\circ h_2\circ\cdots\circ h_n)=\limsup_{r\to\infty}\frac{\log r}{\log T^{-1}_{f_1\circ f_2\circ\cdots\circ f_n\circ g_2}(T_{g_1\circ h_2\circ\cdots\circ h_n}(r))}$$

$$\geq limsup_{r\to\infty}\frac{\log r}{\log(8M^{-1}_{g_2}(18M^{-1}_{f_n}(\cdots 18M^{-1}_{f_2}(9M^{-1}_{f_1}(\exp(3T_{g_1\circ h_2\circ\cdots\circ h_n}(r))))\cdots)))}\text{(from Theorem 3.4)}$$

$$\geq limsup_{r\to\infty}\frac{\log r}{\log(8M^{-1}_{g_2}(18M^{-1}_{f_n}(\cdots \cdots 18M^{-1}_{f_2}(9M^{-1}_{f_1}(\exp[3T_{g_1}(M_{h_2}(M_{h_3}(\cdots M_{h_n}(r)\cdots)))]))\cdots)))}\text{(from Theorem 3.5)}$$

$$\geq limsup_{r\to\infty}\frac{\log r}{\log(8M^{-1}_{g_2}(18M^{-1}_{f_n}(\cdots 18M^{-1}_{f_2}(9M^{-1}_{f_1}(\exp[3T_{g_1}(M_{h_2}(M_{h_3}(\cdots \cdots M_{h_{n-1}}(\exp(r^{\rho_{h_n}+\epsilon}))\cdots)))]))\cdots)))}\text{(from 4.1)}$$

$$\geq limsup_{r\to\infty}\frac{\log r}{\log(8M^{-1}_{g_2}(18M^{-1}_{f_n}(\cdots 18M^{-1}_{f_2}(9M^{-1}_{f_1}(\exp[3T_{g_1}(\exp(\exp(\cdots(\exp(r^{\rho_{h_n}+\epsilon}))\cdots)))^{\rho_{h_2}+\epsilon}]))\cdots)))}$$

$$\geq limsup_{r\to\infty}\frac{\log r}{\log(8M^{-1}_{g_2}(18M^{-1}_{f_n}(\cdots 18M^{-1}_{f_2}(9M^{-1}_{f_1}(\exp[\exp(\exp(\exp(\cdots \cdots(\exp(r^{\rho_{h_n}+\epsilon}))\cdots)))^{\rho_{h_2}+\epsilon}]^{\rho_{g_1}+\epsilon}))\cdots)))}$$

$$\geq limsup_{r\to\infty}\frac{\log r}{\log(8M^{-1}_{g_2}(18M^{-1}_{f_n}(\cdots 18M^{-1}_{f_2}(9M^{-1}_{f_1}(\exp[\exp(\exp(\exp(\cdots(\exp(r^{\rho_{h_n}+\epsilon}))\cdots)))^{\rho_{h_2}+\epsilon}]^{\rho_{g_1}+\epsilon}))\cdots)))}$$

$$\geq limsup_{r\to\infty}\frac{\log r}{\log(8M^{-1}_{g_2}(18M^{-1}_{f_n}(\cdots 18M^{-1}_{f_2}(\exp\{\frac{1}{\lambda_{f_1}-\epsilon}\log^{[2]}(\exp[\exp(\cdots\exp(r^{\rho_{h_n}+\epsilon})\cdots)]^{\rho_{g_1}+\epsilon})\})\cdots)))+O(1)}$$

$$\geq limsup_{r\to\infty}\frac{\log r}{\log(8M^{-1}_{g_2}(18M^{-1}_{f_n}(\cdots 18M^{-1}_{f_2}(\exp\{\frac{\rho_{g_1}+\epsilon}{\lambda_{f_1}-\epsilon}(\exp[\exp(\cdots\exp(r^{\rho_{h_n}+\epsilon})\cdots)^{\rho_{h_2}+\epsilon}])\})\cdots)))+O(1)}$$

$$\geq limsup_{r\to\infty}\frac{\log r}{\log(8M^{-1}_{g_2}(18M^{-1}_{f_n}(\cdots 18M^{-1}_{f_3}(\exp\{\frac{1}{\lambda_{f_2}-\epsilon}\log^{[2]}(\frac{\rho_{g_1}+\epsilon}{\lambda_{f_1}-\epsilon}(\exp[\exp(\cdots \exp(r^{\rho_{h_n}+\epsilon})\cdots)^{\rho_{h_2}+\epsilon}]))\})\cdots)))+O(1)}$$

$$\geq limsup_{r\to\infty}\frac{\log r}{\log(8M^{-1}_{g_2}(18M^{-1}_{f_n}(\cdots 18M^{-1}_{f_3}(\exp\{\frac{\rho_{h_2}+\epsilon}{\lambda_{f_2}-\epsilon}(\exp[\exp(\cdots\exp(r^{\rho_{h_n}+\epsilon})\cdots)^{\rho_{h_3}+\epsilon}])\})\cdots)))+O(1)}$$

$$\vdots$$

$$\geq limsup_{r\to\infty}\frac{\log r}{\log(8M^{-1}_{g_2}(18M^{-1}_{f_n}(\exp\{\frac{\rho_{h_{n-1}}+\epsilon}{\lambda_{f_{n-1}}-\epsilon}r^{\rho_{f_n}+\epsilon}\})))+O(1)}$$

$$\geq limsup_{r\to\infty}\frac{\log r}{\log(8M^{-1}_{g_2}(\exp\{\frac{1}{\lambda_{f_n-\epsilon}}\log^{[2]}(\exp\{\frac{\rho_{h_{n-1}}+\epsilon}{\lambda_{f_{n-1}}-\epsilon}r^{\rho_{f_n}+\epsilon}\})\}))+O(1)}$$

$$\geq limsup_{r\to\infty}\frac{\log r}{\log(8M^{-1}_{g_2}(\exp\{\frac{1}{\lambda_{f_n-\epsilon}}\log\{\frac{\rho_{h_{n-1}}+\epsilon}{\lambda_{f_{n-1}}-\epsilon}r^{\rho_{f_n}+\epsilon}\}\}))+O(1)}$$

$$\geq limsup_{r\to\infty}\frac{\log r}{\log[8M^{-1}_{g_2}(\exp\{\frac{\rho_{f_n}+\epsilon}{\lambda_{f_n}-\epsilon}\log r\})]+O(1)}$$

$$\geq limsup_{r\to\infty}\frac{\log r}{\log(8M^{-1}_{g_2}(r^{\frac{\rho_{f_n}+\epsilon}{\lambda_{f_n}-\epsilon}}))+O(1)}$$

$$\geq limsup_{r\to\infty}\frac{\log r}{\frac{1}{m}\log(\frac{r^{\frac{\rho_{f_n}+\epsilon}{\lambda_{f_n}-\epsilon}}}{|a_m|(1-\epsilon)})+O(1)}\text{(from 4.9)}$$

$$\geq limsup_{r\to\infty}\frac{\log r}{\frac{1}{m}(\log r^{\frac{\rho_{f_n}+\epsilon}{\lambda_{f_n}-\epsilon}}-\log|a_m|(1-\epsilon))+O(1)}$$

$$\geq limsup_{r\to\infty} \frac{\log r}{\frac{1}{m}\frac{\rho_{f_n}+\epsilon}{\lambda_{f_n}-\epsilon}\log r + O(1)}$$
$$\geq m\frac{\lambda_{f_n}-\epsilon}{\rho_{f_n}+\epsilon}$$
$$\geq m\frac{\lambda_{f_n}}{\rho_{h_n}}, \quad \text{as } \epsilon > 0 \text{ is arbitrary.}$$

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