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AN EXTENSION OF HERMITE DIFFERENTIAL EQUATION

DANIEL BRATIANU

Grad Student, Petroleum-Gas University, Ploiesti, Romania
Correspondence: daniel.bratianu@yahoo.com

Abstract: The purpose of this paper is to show that the generating function of Hermite polynomials can be modified so as to lead to a modified Hermite equation and a corresponding solution of modified orthogonal functions. First, by series expansion of the new generating function, the explicit expression of some modified Hermite polynomials and their Rodrigues representation are found. Then, using the recurrence relations, the extended differential equation is deduced, it is shown that the equation is self-adjoint and the orthogonality relation for the solutions of the equation is established.

1. Introduction

Let $F(x,t)$ be the exponential generating function of the Hermite polynomials for all finite x and t, in which we replace the parameter t by $(t-1)$

$$F(x,t) = exp[-(t-1)^2 - 2(t-1)x] \quad (1)$$

After a rearrangement of terms we can rewrite the modified generating function in the form

$$F(x,t) = exp(2x-1)G(x,t) \quad (2)$$

where the function $G(x,t)$ is the generating function of Hermite polynomials in which the variable x is replaced by $(x-1)$

$$G(x,t) = exp[-t^2 - 2(x-1)t] \quad (3)$$

i.e. $G(x,t)$ is the generating function of Hermite polynomials in the case where the polynomials are shifted from the center $c=0$ to the center $c=1$.

2. Extended Hermite Functions

Indeed, according to the Maclaurin's formula, the series expansion of the generating function (3) is

$$\begin{aligned} G(x,t) &= exp[-2(x-1)t]exp(-t^2) \quad (4) \\ &= \left(\sum_{p=0}^{\infty} \frac{(-1)^p[2(x-1)]^p t^p}{p!}\right)\left(\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!}\right) \\ &= \sum_{p=0}^{\infty}\left(\sum_{k=0}^{\infty} \frac{(-1)^{p+k}[2(x-1)]^p}{p!\ k!}\right) t^{p+2k} \end{aligned}$$

Key words and phrases: extended equation; modified functions; self-adjoint; orthogonality property.

If we now collect the powers of t by substituting $p + 2k = n,\ p = n - 2k$, the power series becomes

$$G(x,t) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[n/2]}\frac{(-1)^{n-k}}{k!}\frac{[2(x-1)]^{n-2k}}{(n-2k)!}\right)t^n \quad (5)$$

where $[n/2]$ denotes the integer part of $n/2$. Therefore the generating function of shifted Hermite polynomials (3) can be written as a Maclaurin series in t

$$G(x,t) = \sum_{n=0}^{\infty} P_n(x)t^n \quad (6)$$

where the coefficient of t^n is the polynomial of degree n in the variable x

$$P_n(x) = \sum_{k=0}^{[n/2]}\frac{(-1)^{n-k}}{k!}\frac{[2(x-1)]^{n-2k}}{(n-2k)!} \quad (7)$$

that is the Hermite polynomial centered at $c = 1$. If we also expand the factor $(x-1)^{n-2k}$ according to the binomial theorem, we can rewrite the polynomial as

$$P_n(x) = \sum_{k=0}^{[n/2]}\sum_{q=0}^{n-2k}\frac{(-1)^{n-k+q}\,2^{n-2k}}{k!\ q!}\frac{x^{n-2k-q}}{(n-2k-q)!} \quad (8)$$

Now, by substituting the power series (6) into (2), the extended generating function can be written as a Maclaurin series in t

$$F(x,t) = \sum_{n=0}^{\infty} H_n(x)t^n \quad (9)$$

where the coefficient of t^n is given by

$$H_n(x) = e^{2x-1}P_n(x) \quad (10)$$

which is no longer a polynomial. These are the extended Hermite functions

3. Rodrigues' Formula

Let us rewrite the extended generating function (1) more conveniently as

$$F(x,t) = e^{x^2}e^{-(t+x-1)^2} \quad (11)$$

If we extend this function as a Maclaurin series in t, we get

$$F(x,t) = e^{x^2}\sum_{n=0}^{\infty}\left[\frac{d^n}{dt^n}e^{-(t+x-1)^2}\right]_{t=0}\frac{t^n}{n!} \quad (12)$$

It is easy to observe that we have

$$\frac{d^n}{dt^n}e^{-(t+x-1)^2} = \frac{d^n}{dx^n}e^{-(t+x-1)^2}$$

Therefore, according to the power series (9), the functions $H_n(x)$ get the expression

$$H_n(x) = \frac{e^{x^2}}{n!}\frac{d^n}{dx^n}e^{-(x-1)^2} \quad (13)$$

This is the Rodrigues type formula corresponding to the new generated functions $H_n(x)$

4. Recurrence Relations

By differentiating the extended generating function (1) with respect to t, we get

$$\frac{\partial F(x,t)}{\partial t} = [-2(x-1) - 2t]F(x,t) \quad (14)$$

Replacing the generating function by its series expansion (9), we find

$$\frac{\partial F(x,t)}{\partial t} = \sum_{n=0}^{\infty}(n+1)H_{n+1}(x)t^n \tag{15}$$

Also, we find

$$tF(x,t) = \sum_{n=1}^{\infty} H_{n-1}(x)t^n \tag{16}$$

Thus, by substituting the equations (9), (15) and (16) into (14) we obtain two recurrence relations

$$H_1(x) + 2(x-1)H_0(x) = 0, \text{ if } n = 0 \tag{17}$$

and

$$(n+1)H_{n+1}(x) + 2(x-1)H_n(x) + 2H_{n-1}(x) = 0, \text{ if } n \geq 1 \tag{18}$$

Now, by differentiating (1) with respect to x, we get

$$\frac{\partial F(x,t)}{\partial x} = -2(t-1)F(x,t) \tag{19}$$

Also, by differentiating (9) with respect to x, we get

$$\frac{\partial F(x,t)}{\partial x} = \sum_{n=0}^{\infty} H_n'(x)t^n \tag{20}$$

Thus, by substituting the equations (9), (16) and (20) into (19), we obtain the other two recurrence relations

$$H_0'(x) - 2H_0(x) = 0, \text{ if } n = 0 \tag{21}$$

and

$$H_n'(x) + 2H_{n-1}(x) - 2H_n(x) = 0, \quad \text{if } n \geq 1 \tag{22}$$

5 Extended Hermite Equation

Starting from the recurrence relations (18) and (22) we can derive a modified Hermite equation. Indeed, replacing n by $(n+1)$ into (22) we get the equation

$$H_{n+1}'(x) + 2H_n(x) - 2H_{n+1}(x) = 0 \tag{23}$$

Differentiating (18) with respect to x, we get

$$(n+1)H_{n+1}'(x) + 2H_n(x) + 2(x-1)H_n'(x) + 2H_{n-1}'(x) = 0 \tag{24}$$

Multiplying (23) by $(n+1)$, we get

$$(n+1)H_{n+1}'(x) + 2(n+1)H_n(x) - 2(n+1)H_{n+1}(x) = 0 \tag{25}$$

Subtracting (24) from (25) and simplifying, we find

$$nH_n(x) - (n+1)H_{n+1}(x) - H_{n-1}'(x) - (x-1)H_n'(x) = 0 \tag{26}$$

Adding (18) to (26), we get

$$(n+2x-2)H_n(x) - (x-1)H_n'(x) + 2H_{n-1}(x) - H_{n-1}'(x) = 0 \tag{27}$$

Replacing n by $(n+1)$ into (27), we get

$$(n+2x-1)H_{n+1}(x) - (x-1)H_{n+1}'(x) + 2H_n(x) - H_n'(x) = 0 \tag{28}$$

Adding (28) to $(x-1)$ times (23), we get the equation

$$H_n'(x) - 2xH_n(x) - (n+1)H_{n+1}(x) = 0 \tag{29}$$

Differentiating this equation with respect to x, we get

$$H_n''(x) - 2xH_n'(x) - 2H_n(x) - (n+1)H_{n+1}'(x) = 0 \qquad (30)$$

Adding (25) to (30), we find

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) - 2(n+1)H_{n+1}(x) = 0 \qquad (31)$$

Now, adding (-2) times (29) to (31), we obtain the equation

$$H_n''(x) - 2(x+1)H_n'(x) + 2(n+2x)H_n(x) = 0 \qquad (32)$$

that is the corresponding Hermite equation

6. Orthogonality Relation

It is easy to observe that the modified Hermite differential equation (32) can also be arranged in the following self–adjoint form

$$\frac{d}{dx}\left[r(x)\frac{d}{dx}H_n(x)\right] + [q(x) + \lambda_n w(x)]H_n(x) = 0 \qquad (33)$$

where the coefficient functions $q(x)$, $r(x)$ and $w(x)$ are given by

$$q(x) = 4x\, e^{-(x+1)^2} \qquad (34)$$

$$r(x) = e^{-(x+1)^2} \qquad (35)$$

$$w(x) = r(x) = e^{-(x+1)^2} \qquad (36)$$

and the parameter λ_n is given by $\lambda_n = 2n$. Therefore, the equation (32) is a Sturm-Liouville equation, λ_n are the eigenvalues of the equation and $H_n(x)$ are the eigenfunctions associated with the eigenvalues λ_n. Since we have $r(-\infty) = r(+\infty) = 0$, it follows that the sequence of eigenfunctions $\{H_n(x)\}_{n=0}^{\infty}$ is orthogonal with respect to the weight function $w(x)$ on the interval $[-\infty, +\infty]$, that is

$$\int_{-\infty}^{+\infty} w(x)H_m(x)\, H_n(x)dx = 0, \quad \text{if} \quad m \neq n \qquad (37)$$

Now, let h_n be the integral

$$h_n = \int_{-\infty}^{+\infty} w(x)H_n^2(x)dx \qquad (38)$$

Multiplying the equation (18) by $w(x)H_{n+1}(x)$ and integrating over x from $-\infty$ to $+\infty$, according to the orthogonality property (37) we get the relation

$$(n+1)h_{n+1} + 2\int_{-\infty}^{+\infty}(x-1)w(x)H_n(x)\, H_{n+1}(x)dx = 0 \qquad (39)$$

Now, if we replace n by $(n+1)$ into (18), we get the recurrence relation

$$(n+2)H_{n+2}(x) + 2(x-1)H_{n+1}(x) + 2H_n(x) = 0 \qquad (40)$$

Again, multiplying this relation by $w(x)H_n(x)$ and integrating over x from $-\infty$ to $+\infty$, we get the relation

$$2h_n + 2\int_{-\infty}^{+\infty}(x-1)w(x)H_{n+1}(x)\, H_n(x)dx = 0 \qquad (41)$$

Now, eliminating the integral from (39) and (41) and replacing n by $(n-1)$ we get the recurrence relation

$$h_n = \frac{2}{n}h_{n-1} \qquad (42)$$

Replacing n by $(n-1)$, $(n-2)$, ..., $(n-k)$, we obtain the following formula for h_n in terms of h_{n-k}

$$h_n = 2^k \frac{(n-k)!}{n!} h_{n-k} \tag{43}$$

Now, making $k = n$, we find the relation

$$h_n = \frac{2^n}{n!} h_0 \tag{44}$$

where, according to (7), (10) and (38), h_0 is given by

$$h_0 = e^{-2} \int_{-\infty}^{+\infty} e^{-(x-1)^2}\, dx \tag{45}$$

Thus, using the substitution $\xi = x - 1$, h_0 gets the value

$$h_0 = e^{-2} \int_{-\infty}^{+\infty} e^{-\xi^2}\, d\xi = \frac{\sqrt{\pi}}{e^2} \tag{46}$$

Finally, substituting (46) into (44), we obtain

$$h_n = \frac{2^n}{n!} \frac{\sqrt{\pi}}{e^2} \tag{47}$$

It follows that the sequence of functions $\left\{\widetilde{H}_n(x) = \frac{H_n(x)}{\sqrt{h_n}}\right\}_{n=0}^{\infty}$ is orthonormal with respect to the weight function $w(x)$ on the interval $[-\infty, +\infty]$, that is

$$\int_{-\infty}^{+\infty} w(x) \widetilde{H}_m(x)\, \widetilde{H}_n(x) dx = \delta_{mn} \quad \text{with} \quad \delta_{mn} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases} \tag{48}$$

7. Conclusions

In this paper we have seen that the change of parameter $t \to (t-1)$ into the generating function of Hermite polynomials leads to the change of variable $x \to (x-1)$ into the Hermite polynomials, i.e. it leads to a shift of the Hermite polynomials from the center $c = 0$ to the center $c = 1$. Although, because of the exponential factor e^{2x-1}, the new generated functions are no longer polynomials, they preserve the orthogonality property on the interval $[-\infty, +\infty]$ with respect to the weight function $w(x) = e^{-(x+1)^2}$. This property directly follows from the extended Hermite equation, which is a Sturm-Liouville equation. Therefore, an extension of the Hermite differential equation can be obtained replacing the parameter t by $(t - \alpha)$ into the generating function of Hermite polynomials, where α is any non-zero real number.

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