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CONE METRIC SPACES AND FIXED POINT THEOREMS FOR GENERALIZED T-REICH AND T-RHOADES CONTRACTIVE MAPPINGS

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Abstract. The purpose of this paper is to obtain the generalization of T- Reich and T-Rhoades contractive type mappings on complete cone metric spaces. Our results generalize several well known of recent results.

1. INTRODUCTION

The study of fixed points of mappings satisfying certain contractive conditions has been at the centre of strong research activity. In 2007, Huang Gaung, Zhang Xian [1] generalized the concept of metric space, replacing the set of real numbers by an ordered Banach space, and obtained some fixed point theorems for mappings satisfying different contractive conditions. The result in [1] were generalized by Sh. Rezapour and Hamlbarani in [3] omitting the assumption of normality of the cone. Many authors have studied fixed point theorems in such spaces; see for instance [6],[8], and[10], Recently, A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh [4] introduced a new class of contractive functions T – Contraction and T – Contractive mappings. S. Moradi [7] introduced the notion a T – Kannan contractive mapping which extend the well known Kannan's fixed point theorem [5, 11, and 12]. The result [4] and [7] also generalized by [13], [14] and [15]. In view of these facts, there by the purpose of this paper is to study the existence of fixed point of generalize T – Reich and T-Rhoades contractive mappings defined on a complete cone metric space (X, d) .Our results generalize and extend the respective theorems 3.1, 3.2 of [2].

2. PRELIMINARY NOTES

Definition 2.1 [1]: Let E be a real Branch space and P a subset of E. Then P is called a cone if it is satisfied the following conditions,

(i) P is closed, non-empty and $P \neq \{0\}$;

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(ii) $ax + by \in P$ for all $x, y \in P$ and non negative real numbers $a, b \in R$;

(iii) $x \in P$ *and* $-x \in P \Rightarrow x = 0 \Leftrightarrow P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a Partial ordering $\leq$ on E with respect to P *by* $x \leq y$ if and only if $y - x \in P$. we shall write $x \ll y$ if $y - x \in$ int P, int P denotes the interior of P.

The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $|| x || \leq K || y ||$.

The least positive number satisfying the above is called the normal constant P.

Definition 2.2 [1]: Let X be a non- empty set. Suppose that the mapping $d: X \, x \, X \to E$ satisfies,

(a) $0 < d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;
(b) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(c) $d(x,y) = (x,z) + (y,z)$ for all $x, y \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Example 2.3 [1]: Let $E = R^2$, $P = \{(x,y) \in E: x,y \geq 0\}$, $X = R$ and $d: X \times X \to E$ defined by $d(x,y) = (| x - y |, \propto |x - y|)$ where $\propto \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 2.5[1]: Let (X, d) be a cone metric space. Let $\{\boldsymbol{x}_n\}_{n \geq 1}$ be a sequence in X and $\boldsymbol{x} \in X$. Then

(i) $\{\boldsymbol{x}_n\}_{n \geq 1}$ converges to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(\boldsymbol{x}_n, \boldsymbol{x}) \ll c$ for all $n \geq N$
(ii) $\{\boldsymbol{x}_n\}_{n \geq 1}$ is said to Cauchy sequence for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(\boldsymbol{x}_n, \boldsymbol{x}_m) \ll c$ for *all* $n \, m \geq N$.
(iii) (X, d) is called a complete cone metric space, if every Cauchy sequence is convergent in X.

Lemma 2.4 [1] Let (X, d) be a cone metric space, $P \subset E$ a normal cone with normal constant K.

Let $\{x_n\}$, $\{y_n\}$ be a sequence in X and $x, y \in X$. Then,

(i) $\{x_n\}$ **converges** to x if and only if $\lim_{n \to \infty} d(x_n, x) = 0$;
(ii) **If** $\{x_n\}$ converges to x and $\{x_n\}$ converges to y then $x = y$. That is the limit of

is unique;

(iii) If $\{x_n\}$ converges to x, then $\{x_n\}$ is Cauchy sequence.

(iv) $\{x_n\}$ is a Cauchy sequence if and only if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$;

(v) If $x_n \to x$ and $\{y_n\}$ is another sequence in X such that $y_n \to y$ $(n \to \infty)$ then

$d(x_n, y_n) \to d(x, y)$

Definition 2.6[8] Let (X, d) be a cone metric space, P be a normal cone with normal constant K

and T : X → X then

(i) T is said to be continuous if $\lim_{n\to\infty} x_n = x$ implies that $\lim_{n\to\infty} Tx_n = Tx$ for all $\{x_n\}$ in X;

(ii) T is said to be sub sequentially convergent, if for every sequence $\{y_n\}$ that $\{Ty_n\}$ is Convergent, implies $\{y_n\}$ has a convergent subsequence.

(iii) T is said to be sequentially convergent if for every sequence $\{y_n\}$, if $\{Ty_n\}$ is Convergent,

Then $\{y_n\}$ is also convergent.

Definition 2.7 (see [2]) Let (X, d) be a cone metric space and $T, S: X \to X$ two functions,

(i) A mapping S is said to be a *T*-Reich contraction(TR_1 – Contraction) if there a +b+c < 1

$d(TSx, TSy) \leq ad(Tx, TSx) + bd(Ty, TSy) + cd(Tx, Ty)$ for all $x, y \in X$ and $a, b, c \geq 0$.

(ii) A mapping S is said to be a *T*-Rhoades contraction(TR_2 – Contraction) if there a +b+c < 1

$d(TSx, TSy) \leq ad(Tx, TSy) + bd(Ty, TSx) + cd(Tx, Ty)$ for all $x, y \in X$ and $a, b, c \geq 0$.

3. Main Results

The following results which we will give are generalization of theorem 3.1 and 3.2 of [2].

Theorem 3.1: Let (X, d) be a complete cone metric space, P be a normal cone with norm constant K, In addition let $T: X \longrightarrow X$ be a one to one, continuous function and $R, S: X \longrightarrow X$ be a pair of TR_{1-}contractions. Then

(i) For every $x_0 \in X$,

$$\lim_{n\to\infty} d(TR^{2n+1}x_0, TR^{2n+2}) = 0 \text{ and } \lim_{n\to\infty} d(TS^{2n+2}x_0, TS^{2n+3}) = 0;$$

(ii) There is $v \in X$ such that

$\lim_{n\to\infty} TR^{2n+1} x_0 = v = \lim_{n\to\infty} T S^{2n+2}$;

(iii) If T is subsequently convergent, then $(R^{2n+1}x_0)$ and $(S^{2n+2}x_0)$ have a convergent subsequences;

(iv) There is a unique common fixed point $u \in X$ such that

$$Ru = u = Su;$$

(v) If T is a sequentially convergent, then for each $x_0 \in X$ the iterative sequences $(R^{2n+1}x_0)$ and $(S^{2n+2}x_0)$ convergent to *u.*

Proof: Let x_0 be an arbitrary point in$X$. We define the iterate sequences (x_{2n}) and (x_{2n+1}) by

$$x_{2n+1} = Rx_{2n} = R^{2n}x_0 \text{ and}$$

$$x_{2n+2} = Sx_{2n+1} = S^{2n+1}x_0;$$

Since Rand S are pair of TR1- contractions, we have

$$\begin{aligned} d(Tx_{2n}, Tx_{2n+1}) &= d(TRx_{2n-1}, TRx_{2n}) \\ &\leq ad(Tx_{2n-1}, TRx_{2n-1}) + bd(Tx_{2n}, TRx_{2n}) + cd(Tx_{2n-1}, Tx_{2n}) \\ &\leq ad(Tx_{2n-1}, Tx_{2n}) + bd(Tx_{2n}, Tx_{2n+1}) + cd(Tx_{2n-1}, Tx_{2n}) \end{aligned}$$

Similarly

$$\begin{aligned} d(Tx_{2n+1}, Tx_{2n+2}) &= d(TSx_{2n}, TSx_{2n+1}) \\ &\leq a'd(Tx_{2n}, TSx_{2n}) + b'd(Tx_{2n+1}, TSx_{2n+1}) + c'd(Tx_{2n}, TSx_{2n}) \\ &\leq a'd(Tx_{2n}, Tx_{2n+1}) + b'd(Tx_{2n+1}, Tx_{2n+1}) + c'd(Tx_{2n}, Tx_{2n+1}) \end{aligned}$$

So, $d(Tx_{2n}, Tx_{2n+1}) \leq \left(\frac{a+c}{1-b}\right) d(Tx_{2n-1}, Tx_{2n})$ and

$$d(Tx_{2n+1}, Tx_{2n+2}) \leq \left(\frac{a'+c'}{1-b'}\right) d(Tx_{2n}, Tx_{2n+1})$$

We can conclude, by repeating the same argument, that

$$d(TR^{2n}x_0, TR^{2n+1}x_0) \leq \left(\frac{a+c}{1-b}\right)^{2n} d(Tx_0, TRx_0) \text{..............................} (3.1)$$

And $d(TS^{2n+1}x_0, TS^{2n+2}x_0) \leq \left(\frac{a'+c'}{1-b'}\right)^{2n+1} d(Tx_0, TSx_0)$........................... (3.2)

From(3.1) we have

$$\|(d(TR^{2n}x_0,TR^{2n+1}x_0))\| \leq \left(\frac{a+c}{1-b}\right)^{2n} K \left\|\left(d\left(Tx_{0,}TRx_0\right)\right)\right\|$$

Where $K$is the normal constant of E. By inequality above we get

$\lim_{n\to\infty}\|d(TR^{2n}x_0,TR^{2n+1}x_0)\| = 0$

Hence,

$$\lim_{n\to\infty} d(TR^{2n}x_0,TR^{2n+1}x_0) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.3)$$

By (3.1), for $m,n \in N$with $m > n$, we have

$$d(Tx_{2m},Tx_{2n}) \leq d\left(Tx_{2n,}Tx_{2n+1}\right) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + d(Tx_{2m-1},Tx_{2m})$$

$$\leq [\left(\frac{a+c}{1-b}\right)^{2n} + \ldots\ldots\ldots\ldots\ldots\ldots + \left(\frac{a+c}{1-b}\right)^{2m-1}]\ d(Tx_0,TRx_0)$$

$$= \left(\frac{a+c}{1-b}\right)^{2n} \mathrm{X} \frac{1}{1-\frac{a+c}{1-b}}\ d(Tx_0TRx_0)$$

$$d(TR^{2n}x_0,TR^{2m}x_0) \leq \left(\frac{a+c}{1-b}\right)^{2n} \mathrm{X} \frac{1}{1-\frac{a+c}{1-b}} d(Tx_0TRx_0) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.4)$$

From (3.4) we have

$$\|d(TR^{2n}x_0,TR^{2m}x_0)\| \leq \left(\frac{a+c}{1-b}\right)^{2n} \mathrm{X} \frac{K}{1-\frac{a+c}{1-b}} \|d(Tx_0TRx_0)\|$$

Where K is the normal constant of E. Taking limit and by$\frac{a+c}{1-b} < 1$, we obtain

$\lim_{n,m\to\infty}\|d(TR^{2n}x_0,TR^{2m}x_0)\| = 0$.In this way, we have

$\lim_{n\to\infty} d(TR^{2n}x_0,TR^{2m}x_0) = 0$, which implies that $(TR^{2n}x_0)$ is a Cauchy sequence in X. Since X is a complete cone metric space, then there is $v \in X$such that

$$\lim_{n\to\infty} TR^{2n}\, x_0 = v \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.5)$$

Now, if T is sub sequentially $\{R^{2n}x_0\}$ has a convergent sub sequence. So there are$u \in X$and $\{x_{2n(i)}\}$ such that

$$\lim_{i\to\infty} R^{2n(i)}\, x_0 = u \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . (3.6)$$

Since T is continuous and by (3.6) we obtain

$$\lim_{i\to\infty} TR^{2n(i)}x_0 = \mathrm{T}\,u \text{ .... } (3.7)$$

By (3.5) and (3.7) we conclude that

$$Tu = v\text{.................................... .... } (3.8)$$

On the other hand,

$$d(TRu.Tu) \leq d\big(TRu, TR^{2n(i)}x_0\big) + d\big(TR^{2n(i)}x_0, TR^{2n(i)+1}x_0\big) + d\big(TR^{2n(i)+1}x_0, Tu\big)$$

$$\leq ad(Tu,TRu)bd\big(TR^{2n(i)-1}x_0, TR^{2n(i)}x_0\big)cd\big(TR^{2n(i)-1}x_0, TR^{2n(i)}x_0\big)$$

$$+\left(\frac{a+c}{1-b}\right)^{2ni} d\big(Tx_{0,}TRx_0\big) + d\big(TR^{2n(i)+1}x_0, Tu\big).$$

Hence,

$$(1-a)d(TRu,Tu) \leq bd\big[\big(TR^{2n(i)-1}x_0, TR^{2n(i)}x_0\big) + cd\big(Tu, TR^{2n(i)-1}x_0\big)\big]$$

$$+\left(\frac{a+c}{1-b}\right)^{2ni} d\big(Tx_{0,}TRx_0\big) + d\big(TR^{2n(i)+1}x_0, Tu\big).$$

$$d(TRu,Tu) \leq \frac{b}{1-a}d\,\big(TR^{2n(i)-1}x_0, TR^{2n(i)}x_0\big) + \frac{c}{1-a}d\big(Tu, TR^{2n(i)-1}x_0\big)$$

$$+\frac{1}{1-a}\left(\frac{a+c}{1-b}\right)^{2ni} d\big(Tx_{0,}TRx_0\big) + d\big(Tx_{0,}TRx_0\big) + \frac{1}{1-a}d\big(TR^{2n(i)+1}x_0, Tu\big).$$

Thus,

$$\|d(TRu,Tu)\| \leq \frac{bK}{1-a}\big\|d\big(TR^{2n(i)-1}x_0, TR^{2n(i)}x_0\big)\big\| + \frac{cK}{1-a}\big\|d\big(Tu, TR^{2n(i)-1}x_0\big)\big\|$$

$$+\frac{1}{1-a}\left(\frac{a+c}{1-b}\right)^{2ni} K\big\|d\,\big(TRx_{0,}Tx_0\big)\big\| + \frac{1}{1-a}K\big\|d\,\big(TR^{2n(i)+1}x_0, Tu\big)\big\| \longrightarrow 0 \text{as i} \longrightarrow \infty.$$

Where K is the normal constant of X. Hence $d(TRu,Tu) = 0$, which implies that $TRu = Tu$. Since T is one to one, then $Ru = u$. consequently R has a fixed point because R is a TR_1-contraction. We get

$$d(TRu,TRv) \leq a[d(Tu,TRu)] + b[d(Tv,TRv)] + c\,[d(Tu,Tv)].$$

If v is another fixed point of S, then from injective of T, we get $Ru = Rv$,or which is the same, the fixed point is unique. Finally, if T is sequentially convergent, by replacing $2n$ for $2ni$ we conclude that

$\lim_{n\to\infty} R^{2n}x_0 = u$. This shows that $\{R^{2n}x_0\}$ converges to the fixed point of R.

Similarly we can prove that $\{S^{2n+1}x_0\}$ converges to the fixed point of S.

$$i.e.\ \lim_{n\to\infty} R^{2n}x_0 = u = \lim_{n\to\infty} S^{2n+1}x_0$$

This completes the proof of the theorem.

Theorem 3.2: Let (X, d) be a complete cone metric space, P be a normal cone with norm constant K, In addition let $T: X \longrightarrow X$ be a one to one, continuous function and $R, S: X \longrightarrow X$ be a pair of TR_2_contractions. Then (1), (2), (3), (4) and (5) of theorem 3.1 hold.

Proof: Let x_0 be an arbitrary point in$X$. We define the iterate sequences (x_{2n}) and (x_{2n+1}) by

$$x_{2n+1} = Rx_{2n} = R^{2n}x_0 \text{ and}$$

$$x_{2n+2} = Sx_{2n+1} = S^{2n+1}x_0;$$

Since R and S are pair of TR_2 – contraction, we have

$$d(TRx_{2n}, TRx_{2n+1}) \le a\, d(Tx_{2n}, TRx_{2n+1}) + b\, d(Tx_{2n+1}, TRx_{2n}) + c\, d(Tx_{2n}, Tx_{2n+1})$$

$$\le a\, d(TRx_{2n-1}, TRx_{2n+1}) + bd(TRx_{2n}, TRx_{2n})c\, d(TRx_{2n-1}, TRx_{2n}) \text{or}$$

$$\le \quad a\{d(TRx_{2n-1}, TRx_{2n}) + d(TRx_{2n}, TRx_{2n+1})\} + c\, d(TRx_{2n-1}, TRx_{2n})$$

$$d(TRx_{2n}, TRx_{2n+1}) \le \left(\tfrac{a+c}{1-a}\right)\ d(TRx_{2n-1}, TRx_{2n})$$

$$= h\, d(TRx_{2n-1}, TRx_{2n}) \text{ where } h = \left(\tfrac{a+c}{1-a}\right).$$ Recursively, we obtain

$$d(TRx_{2n}, TRx_{2n+1}) \le h^{2n}\ d(TRx_o, TRx_1)\text{...}(3.9)$$

Therefore

$$\|d(TRx_{2n}, TRx_{2n+1})\| \le h^{2n}k\|d(TRx_o, TRx_1)\|,$$

where K is the normal constant of X. Hence $\lim_{n\to\infty}\|d(TRx_{2n}, TRx_{2n+1})\| = 0$.

This implies that $\lim_{n\to\infty} d(TR^{2n}x_0, TR^{2n+1}x_0) = 0$.

Similarly, we have

$$\lim_{n\to\infty} d(TS^{2n+1}x_0, TS^{2n+2}x_0) = 0.$$

By (3.9), for every $m, n \in N$ with $n > m, we\ have,$

$$d(TRx_{2m}, TRx_{2n}) \leq d(TRx_{2n,} TRx_{2n+1}) + \ldots\ldots\ldots\ldots\ldots\ldots + d(TRx_{2m-1}, TRx_{2m})$$

$$\leq [h^{2n-1} + h^{2n-2} + \cdots \ldots \ldots \ldots \ldots \ldots + h^{2m}]\, d(TRx_0, TRx_1)$$

$$\leq \frac{h^{2m}}{1-h}\, d(TRx_0, TRx_1).$$

Taking norm we get

$$\|d(TRx_0, TRx_1)\| \leq \frac{h^{2m}}{1-h} K\|d(TRx_0, TRx_1)\|,$$

Consequently, we have

$$\lim_{n,m\to\infty} d(TRx_0, TRx_1) = 0.$$

Hence $\{TR^{2n}x_0\}$ is a Cauchy sequence in X and since (X, d) is a complete cone metric space, there is $v \in X$ such that

$$\lim_{n\to\infty} d(TR^{2n}x_0) = v.$$

Similarly, we can prove that $\lim_{n\to\infty} d(TS^{2n+1}x_0) = v$. The rest of the proof is similar to the proof of theorem 3.1.

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