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THE MC-DAGUM DISTRIBUTION AND ITS STATISTICAL PROPERTIES WITH APPLICATIONS

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Abstract. In this paper, a new class of distributions called Mc-Dagum distribution is proposed. This class of distributions contains several distributions such as beta-Dagum, beta-Burr III, beta-Fisk, Dagum, Burr III and Fisk distributions as special cases. The hazard function, reverse hazard function, moments, mean residual life function, Renyi entropy and Fisher information are obtained. Lorenz, Bonferroni and Zenga curves are derived. Maximum likelihood estimates of the model parameters and numerical examples are given to illustrate the usefulness of the proposed class of distributions.

1. Introduction

Camilo Dagum [1] proposed the Dagum distribution to fit empirical income and wealth data, and also accommodate heavy tailed models. Dagum distribution has both Type-I and Type-II specification, where Type-I is the three parameter specifications and Type-II deal with four parameter specification. The cumulative distribution function (cdf) and probability density function (pdf) of the three-parameter Dagum distribution are given by

$$G(x;\lambda,\delta,\beta)=\left(1+\lambda x^{-\delta}\right)^{-\beta}, \tag{1}$$

and

$$g\left(x;\lambda,\delta,\beta\right)=\beta\lambda\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta-1}, \quad \text{for } \lambda,\ \delta,\ \beta>0, \tag{2}$$

respectively, where λ is a scale parameter, and δ and β are shape parameters. Dagum refers to his model as the generalized logistic-Burr distribution. Actually, when $\beta=1$, Dagum distribution is referred to as the log-logistic distribution [13]. The most popular Burr distributions are Burr-XLL distribution, often called Burr distribution with cdf,

$$F\left(x;\delta,\beta\right)=1-\left(1+x^{-\delta}\right)^{-\beta}, \quad \text{for } x>0,\ \delta,\ \beta>0, \tag{3}$$

and more importantly the Burr-III distribution with cdf

$$F\left(x;\delta,\beta\right)=\left(1+x^{-\delta}\right)^{-\beta}, \quad \text{for } x>0 \text{ and } \delta,\ \beta>0. \tag{4}$$

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Dagum distribution is a Burr III distribution with an additional scale parameter (λ), [13]. The k^{th} moment of Dagum distribution is given by

$$E\left(X^k\right) = \beta\lambda^{\frac{k}{\delta}} B\left(\beta+\frac{k}{\delta}, 1-\frac{k}{\delta}\right), \tag{5}$$

for $\delta > k, \lambda, \delta, \beta > 0$, where B(.,.) is the beta function, (by setting $t = (1+\lambda x^{-\delta})^{-1}$). The q^{th} percentile of the Dagum distribution is

$$x(q) = \lambda^{\frac{1}{\delta}}(q^{\frac{-1}{\beta}} - 1)^{\frac{-1}{\delta}}. \tag{6}$$

Additional results on the Dagum distribution, including estimation of the parameters for censored data and properties of the beta-Dagum distribution can be seen in McDonald and Xu [14], Domma and Condino [2] and Domma et al. [3].

1.1. **Basic Utility Notions.** Some useful functions that are employed in subsequent sections are given below. The gamma and digamma functions are given by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$ and $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ respectively, where $\Gamma'(x) = \int_0^\infty t^{x-1}(\log t)e^{-t}\,dt$ is the first derivative of the gamma function. The n^{th}-order derivative formula of gamma function is given by: $\Gamma^{(n)}(s) = \int_0^\infty z^{s-1}(\log z)^n \exp(-z)\,dz$. The lower incomplete gamma function and the upper incomplete gamma function are

$$\gamma(s,x) = \int_0^x t^{s-1}e^{-t}\,dt \quad \text{and} \quad \Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}\,dt, \tag{7}$$

respectively.

1.2. **Mc-Donald Generalized Distribution.** Consider an arbitrary parent cdf $G(x)$. The pdf $f(x)$ of the Mc-Donald generalized distribution is given by

$$f(x;a,b,c) = \frac{cg(x)}{B(a,b)}G^{ac-1}(x)\left(1-G^c(x)\right)^{b-1}, \quad \text{for } a>0,\ b>0, \text{ and } c>0. \tag{8}$$

Note that $g(x)$ is the pdf of parent distribution , $g(x) = dG(x)/dx$, and a,b and c are additional shape parameters. Introduction of this additional shape parameters is specially to introduce skewness. Also, this allows us to vary tail weight. Cordeiro et al. [6] presented results on the McNormal distribution. Marciano et al. [4] obtained the statistical properties of the Mc-Γ distribution and applied the model to reliability data. It is important to note that for c=1 we obtain a sub-model of this generalization which is a beta-generalization and for $a = 1$, we have the Kumaraswamy (Kum) generalized distributions. If a random variable X has the pdf above, we write $X \sim$ Mc-G(a,b,c). The cdf for this generalized distribution is given by

$$F(x;a,b,c) = I_{G(x)^c}(a,b) = \frac{1}{B(a,b)}\int_0^{G(x)^c} \omega^{a-1}(1-\omega)^{b-1}d\omega, \tag{9}$$

where $I_x(a,b) = B(a,b)^{-1}\int_0^{G(x)^c} \omega^{a-1}(1-\omega)^{b-1}d\omega$ denotes the incomplete beta function ratio (Gradshteyn and Ryzhik, [8]). The same equation can be expressed as follows:

$$F(x;a,b,c) = \frac{G\left(x\right)^{ac}}{aB\left(a,b\right)}\left[{}_2F_1\left(a,1-b;a+1;G(x)^c\right)\right], \tag{10}$$

where

$$ {}_2F_1\left(a,b;c;x\right) = B\left(b,c-b\right)^{-1}\int_0^1 \frac{t^{b-1}\left(1-t\right)^{c-b-1}}{\left(1-tz\right)^{a}}dt, \tag{11}$$

is the well known hypergeometric function (Gradshteyn and Ryzhik, [8]), and $B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

An important motivation for the development of the McDonald-Dagum (Mc-D) distribution is the benefit of this class in its ability to fit skewed data that cannot properly be fitted in many other existing distributions. Mc-G family of densities allows for higher levels of flexibility of its tails and has a lot of applications in various fields. See Cordeiro el al. [5] for additional results on generalized distributions and McDonald Weibull distribution [7].

The hazard and reverse hazard functions for the Mc-G distribution are given by

$$ h_F\left(x\right) = \frac{f(x;a,b,c)}{1-F(x;a,b,c)} = \frac{cg\left(x\right)G^{ac-1}\left(x\right)\left\{1-G^{c}\left(x\right)\right\}^{b-1}}{B\left(a,b\right)\left\{1-I_{G(x)^c}\left(a,b\right)\right\}}, \tag{12}$$

and

$$ \tau_F\left(x\right) = \frac{f(x;a,b,c)}{F(x;a,b,c)} = \frac{cg\left(x\right)G^{ac-1}\left(x\right)\left\{1-G^{c}\left(x\right)\right\}^{b-1}}{B\left(a,b\right)I_{G^c(x)}(a,b)}, \tag{13}$$

respectively, for $a>0$, $b>0$, and $c>0$.

The outline of this paper is as follows. In section 2, the Mc-Dagum distribution and related family of distributions are introduced. The series expansion for the density, hazard and reverse hazard functions, and other properties are presented in section 3. Section 4 contains the moments, Lorenz, Bonferroni and Zenga curves. Section 5 deals with measures of uncertainty, including Renyi entropy. Section 6 contains maximum likelihood estimates of the model parameters. Fisher information and asymptotic confidence intervals are presented in section 7, followed by applications in section 8. Some concluding remarks are given in section 9.

2. Mc-Dagum Distribution

In this section, the new class of distributions, called McDonald-Dagum (Mc-D) distribution is introduced. Considering the properties and some useful features of both Dagum and Mc-Donald distributions, a broad range of generalization is possible by combining these distributions. The new class of distributions possess capabilities widely applicable in several areas including economics, finance and reliability.

Now, combining the Mc-G and Dagum distributions, we obtain the pdf of the Mc-Dagum distribution as follows:

$$ f(x;\lambda,\delta,\beta,a,b,c) \quad = \quad \frac{c\beta\lambda\delta x^{-\delta-1}}{B(a,b)}\left(1+\lambda x^{-\delta}\right)^{-\beta ac-1}\left[1-\left(1+\lambda x^{-\delta}\right)^{-c\beta}\right]^{b-1}, $$

for $a,b,c,\lambda,\beta,\delta>0$. The cdf of this new distribution is given by

$$ F(x) \quad = I_{(1+\lambda x^{-\delta})^{-\beta c}}\left(a,b\right), \tag{14}$$

where

$$ I_y(a,b) = \frac{1}{B(a,b)}\int_0^y \omega^{a-1}(1-\omega)^{b-1}d\omega $$

is the incomplete beta function. The cdf can also be written as follows:

$$F(x)=\frac{\left(1+\lambda x^{-\delta}\right)^{-\beta ac}}{aB(a,b)}\left[{}_2F_1\left(a,1-b;a+1;(1+\lambda x^{-\delta})^{-\beta c}\right)\right], \tag{15}$$

where

$${}_2F_1\left(a,b;c;x\right)=\frac{1}{B(b,c-b)}\int_0^1\frac{y^{b-1}(1-y)^{c-b-1}}{(1-y^z)^a}dy, \tag{16}$$

is the well-known hypergeometric function, (Gradshteyn and Ryzhik,[8]).

The hazard and reverse hazard functions are given by

$$h_F\left(x;\lambda,\delta,\beta,a,b,c\right) = \frac{c\beta\lambda\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta ac-1}\left[1-(1+\lambda x^{-\delta})^{-c\beta}\right]^{b-1}}{B(a,b)\left[1-I_{\left[\left(1+\lambda x^{-\delta}\right)^{-\beta c}\right]}(a,b)\right]},$$

and

$$\tau_F\left(x;\lambda,\delta,\beta,a,b,c\right) = \frac{c\beta\lambda x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta ac-1}\left[1-\left(1+\lambda x^{-\delta}\right)^{-c\beta}\right]^{b-1}}{B(a,b)I_{\left(1+\lambda x^{-\delta}\right)^{-\beta c}}(a,b)}$$

for $a>0,b>0,c>0,\lambda>0,\beta>0,\delta>0$, respectively.

3. Expansion of Distribution

Note that for $|\omega|<1$, and $b>0$, $(1-\omega)^{b-1}=\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(b)}{\Gamma(b-j)j!}\omega^j$. Therefore, the cdf can be expanded to obtain

$$\begin{aligned} F(x;\lambda,\beta,\delta,a,b,c) &= \frac{1}{B(a,b)}\int_0^{\left(1+\lambda x^{-\delta}\right)^{-\beta c}}\omega^{a-1}\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(b)}{\Gamma(b-j)j!}d\omega \\ &= \sum_{j=0}^{\infty}p_jG(x;\lambda,\beta c(a+j),\delta), \end{aligned} \tag{17}$$

where $p_j=\frac{(-1)^j\Gamma(b)}{B(a,b)\Gamma(b-j)j!(a+j)}$. Similarly, the pdf is given by

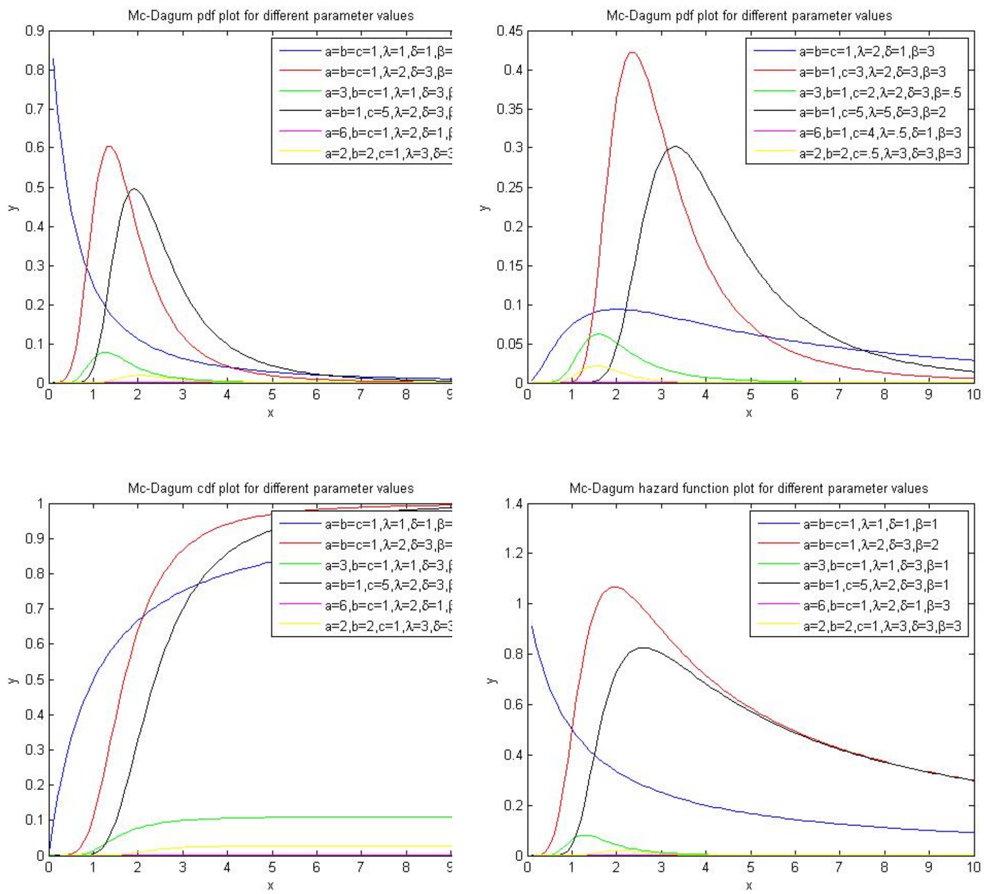
$$f(x)=\sum_{j=0}^{\infty}p_jg(x;\lambda,\beta c(a+j),\delta). \tag{18}$$

If b is an integer, then the summation in equations (17) and (18) stops at $b-1$.

If $c=1$, we have a finite mixture of Dagum distribution with pdf

$$f(x;\lambda,\beta,\delta,a,b)=\sum_{j=0}^{b-1}p_jg(x;\beta\left(a+j\right),\lambda,\delta).$$

The graphs below are the pdf, cdf and hazard function of the Mc-Dagum distribution for different values of parameters λ,δ,β,a,b, and c. The graphs of hazard function shows various shapes including decreasing, unimodal shapes for the selected values of the model parameters.



3.1. **Submodels.** With this generalization, we have several sub-models that can be obtained with specific values of parameters $\lambda, \beta, \delta, a, b$ and c.

(1) When $c = 1$, the Mc-Dagum distribution is the beta-Dagum Distribution, with the density given by:

$$f(x;\lambda,\beta,\delta,a,b) = \frac{\beta\lambda\delta x^{-\delta-1}}{B(a,b)}\left(1+\lambda x^{-\delta}\right)^{-\beta a-1}\left[1-(1+\lambda x^{-\delta})^{-\beta}\right]^{b-1}, \tag{19}$$

for $x > 0$, $\lambda > 0$, $\beta > 0$, $\delta > 0$, $a > 0$, and $b > 0$. See Domma and Condino [3] for additional details.

(2) If $a = b = c = 1$, we have the Dagum distribution with the pdf,

$$f_D\left(x;\lambda,\delta,\beta\right) = \beta\lambda\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta-1}, \tag{20}$$

for $\lambda, \delta, \beta > 0$.

(3) If $b = c = 1$ and $a > 0$, then we have the Dagum distribution with parameters $\beta a, \lambda$ and δ. The pdf is

$$f\left(x;\beta a,\lambda,\delta,\right) = \beta a\lambda\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta a-1}, \tag{21}$$

for $\lambda, \delta, \beta > 0$.

(4) If $a = c = 1$ and $b > 0$,we have another Beta-Dagum distribution with parameters $b, \beta, \lambda, \delta$ and the pdf is given by

$$f_{BD}\left(x;\lambda,\delta,\beta,b\right) = b\beta\lambda\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta-1}\left[1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right]^{b-1}, \tag{22}$$

for $\lambda, \delta, \beta > 0$.

(5) If $a = c = \lambda = 1$, then we have the beta-Burr III distribution with parameters b, β, δ and the pdf is given by

$$f_{BB}\left(x;\delta,\beta,b\right) = b\beta\delta x^{-\delta-1}\left(1+x^{-\delta}\right)^{-\beta-1}\left[1-\left(1+x^{-\delta}\right)^{-\beta}\right]^{b-1}, \tag{23}$$

for $b, \delta, \beta > 0$.

(6) If $c = \beta = 1$, then we have the beta-Fisk distribution with parameters a, b, λ, δ and the pdf is given by

$$f_{BF}\left(x;\lambda,\delta,a,b\right) = \frac{\lambda\delta x^{-\delta-1}}{B\left(a,b\right)}\left(1+\lambda x^{-\delta}\right)^{-a-1}\left[1-\left(1+\lambda x^{-\delta}\right)^{-1}\right]^{b-1}, \tag{24}$$

for $a, b, \lambda, \delta > 0$.

3.2. **Kum-Dagum Distribution.** Kumaraswamy [11] in his paper (1980) proposed a two-parameter distribution (Kumaraswamy distribution) defined in (0, 1). Here we will refer to it as Kum distribution. Its cdf is given by:

$$F(x;a;b) = 1-(1-x^a)^b, \quad x \in (0,1),\ a>0,\ b>0.$$

The parameters a and b are the shape parameters. The Kum distribution has the pdf given by:

$$f(x;a,b) = abx^{a-1}(1-x^a)^{b-1}, \quad x \in (0,1),\ a>0,\ b>0.$$

Note that the Kumaraswamy distribution can be derived from the beta distribution. Combining cdf of Kumaraswamy distribution with the Dagum distribution discussed in earlier, we obtain Kum-Dagum distribution with the cdf and pdf given by

$$F_{Kum}(x) = 1-\left[1-\left(1+\lambda x^{-\delta}\right)^{-\beta a}\right]^{b}$$

and

$$f_{Kum}(x) \quad = \quad ab\beta\lambda\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta a-1}\left(1-\left[1+\lambda x^{-\delta}\right]^{-\beta a}\right)^{-\beta-1},$$

for $a, b, \beta, \lambda, \delta > 0$, respectively. We do not study the properties of the Kum-Dagum distribution in this paper.

4. Moments and Income Inequality Measures

In this section, we present moments, Lorenz, Bonferroni and Zenga curves for the Mc-Dagum distribution. Income distribution and its variation is an important concern for economists. We use the results presented earlier, which was obtained by expanding the pdf.

4.1. **Moments.** We can derive the k^{th} moment of a Mc-D distribution using properties of the mixture distribution. The k^{th} raw or non-central moments are given by

(25)
$$\begin{aligned} E(X^k) &= \int_0^\infty x^k \frac{c\beta\lambda x^{-\delta-1}}{B(a,b)} \left(1+\lambda x^{-\delta}\right)^{-\beta ac-1} \left(1-\left(1+\lambda x^{-\delta}\right)^{\beta c}\right)^{b-1} dx \\ &= \frac{c\beta\lambda}{B(a,b)} \int_0^\infty x^{k-\delta-1} \left(1+\lambda x^{\delta}\right)^{-\beta ac-1} \left(1-\left(1+\lambda x^{-\delta}\right)^{-\beta c}\right)^{b-1} dx. \end{aligned}$$

Now let $y^{-1} = \left(1+\lambda x^{-\delta}\right)$ then $x = (1-y)^{\frac{-1}{\delta}}(\lambda y)^{\frac{1}{\delta}}$, and we have

$$E(X^k) = \frac{c\beta}{\delta B(a,b)} \int_0^1 (1-y)^{\frac{-k}{\delta}} (\lambda y)^{\frac{k}{\delta}} y^{\beta ac-1} (1-y^{\beta c})^{b-1} dy.$$

Using the fact that $(1-y^{\beta c})^{b-1} = \sum_{j=1}^\infty \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)j!}(y^{\beta c})^j$, and for $p_j = \frac{(-1)^j \Gamma(a+b)}{j!\Gamma(a)\Gamma(b-j)(a+j)}$, we obtain

$$\begin{aligned} E(X^k) &= \frac{\lambda^{\frac{k}{\delta}} c\beta}{\delta B(a,b)} \sum_0^\infty \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)j!} \int_0^1 y^{\frac{k}{\delta}+\beta ac+\beta cj-1} (1-y)^{1-\frac{k}{\delta}-1} dy \\ &= \frac{\lambda^{\frac{k}{\delta}} c\beta}{\delta B(a,b)} \sum_0^\infty \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)j!} B(\beta c(a+j)+\tfrac{k}{\delta}, 1-\tfrac{k}{\delta}) \\ &= \sum_{j=0}^\infty \frac{p_j \beta c(a+j)\lambda^{\frac{k}{\delta}}}{\delta} B(\beta c(a+j)+\tfrac{k}{\delta}, 1-\tfrac{k}{\delta}), \end{aligned}$$

for $\delta > k$.

The mean residual life function (MRLF) denoted by $\mu(x;\lambda,\beta,\delta,a,b,c) = \mu(x)$ is given by

(26)
$$\begin{aligned} \mu(x) &= E[X-x|X \geq x] \\ &= \frac{E(X)-E(X|X\leq x)}{1-F(x)} - x \\ &= \frac{\sum_{j=0}^\infty \frac{p_j\beta c(a+j)\lambda^{\frac{k}{\delta}}}{\delta} B(\beta c(a+j)+\frac{k}{\delta},1-\frac{k}{\delta}) - \sum_0^x p_j \frac{\beta c(a+j)\lambda^{\frac{k}{\delta}}}{\delta} B((1+\lambda x^{-\delta})^{-1};\beta c(a+j)+\frac{k}{\delta},1-\frac{k}{\delta})}{1-\sum_{j=0}^\infty p_j G(x;\lambda,\beta c(a+j),\delta)} - x. \end{aligned}$$

4.2. **Lorenz, Bonferroni and Zenga Curves.** Lorenz, Bonferroni and Zenga curves are the most widely used inequality measures in income and wealth distribution, [9]. In this section, we will derive Lorenz, Bonferroni and Zenga curves for the Mc-Dagum distribution. The Lorenz, Bonferroni and Zenga curves are defined by

$$L_F(x) = \frac{\int_0^x tf(t)\,dt}{E(X)} \quad \text{and} \quad B(F(x)) = \frac{\int_0^x tf(t)dt}{F(x)E(X)}$$

and $A(x) = 1-\frac{\mu^-(x)}{\mu^+(x)}$, respectively, where $\mu^-(x) = \frac{\int_0^x tf(t)dt}{F(x)}$ and $\mu^+(x) = \frac{\int_0^x f(t)dt}{1-F(x)} = \frac{E(X)-E_{X>x}(X)}{1-F(x)}$ are the lower and upper means. For Mc-Dagum distribution, using these results, we obtain the curves. Lorenz and Bonferroni curves for Mc-Dagum distribution are given by

$$L_{F_G}(x;\Theta) = \frac{\sum_{j=0}^x p_j\beta c(a+j)\lambda^{\frac{1}{\delta}} B((1+\lambda x^{-\delta})^{-1};\beta c(a+j)+\frac{1}{\delta},1-\frac{1}{\delta})}{\sum_{j=0}^\infty p_j\beta c(a+j)\lambda^{\frac{1}{\delta}} B(\beta c(a+j)+\frac{1}{\delta},1-\frac{1}{\delta})}, \tag{27}$$

and

(28)
$$B(F_G(x;\Theta)) = \frac{\sum_{j=0}^x p_j\beta c(a+j)\lambda^{\frac{1}{\delta}} B((1+\lambda x^{-\delta})^{-1};\beta c(a+j)+\frac{1}{\delta},1-\frac{1}{\delta})}{\sum_{j=0}^\infty p_j G(x;\lambda,\beta c(a+j),\delta) \sum_{j=0}^\infty p_j\beta c(a+j)\lambda^{\frac{1}{\delta}} B(\beta c(a+j)+\frac{1}{\delta},1-\frac{1}{\delta})},$$

respectively, where $\Theta = (\lambda, \beta, \delta, a, b, c)$. Zenga curve for the Mc-Dagum distribution is given by

$$A(x;\Theta) = 1 - \frac{(1-F(x))E[X|X\leq x]}{F(x)[E(X)-E(X|X\leq x)]}, \tag{29}$$

where $E[X|X \leq x] = \sum_0^x p_j \frac{\beta c(a+j)\lambda^{\frac{1}{\delta}}}{\delta} B((1+\lambda x^{-\delta})^{-1}; \beta c(a+j)+\frac{1}{\delta}, 1-\frac{1}{\delta})$, $E(X) = \frac{\sum_{j=0}^{\infty} p_j \beta c(a+j)\lambda^{\frac{1}{\delta}}}{\delta} B(\beta c(a+j)+\frac{1}{\delta}, 1-\frac{1}{\delta})$, and $F(x) = \sum_{j=0}^{\infty} p_j G(x;\lambda,\beta c(a+j),\delta)$.

5. Measures of Uncertainty

In this section, we discuss the Renyi entropy [15], Shannon entropy [16] and $\tilde{\beta}$-entropy for the Mc-Dagum distribution.

5.1. **Renyi and Shannon Entropy.**

For a pdf $f(x)$, the Renyi entropy is given by

$$H_R(f) = \frac{log}{1-s}\left(\int_0^\infty f^s(x)dx\right), \quad s > 0, \text{ and } s \neq 1. \tag{30}$$

As $s \to 1$, we obtain the Shanon entropy. Note that,

$$f^s(x) = \frac{(c\beta\lambda\delta)^s x^{-s\delta-s}}{B^s(a,b)}\left(1+\lambda x^{-\delta}\right)^{-\beta acs-s}\left[1-\left(1+\lambda x^{-\delta}\right)^{-c\beta}\right]^{bs-s}$$

and

$$\begin{aligned}
\int_0^\infty f^s(x)dx &= \frac{(c\beta\lambda\delta)^s}{B^s(a,b)}\int_0^\infty x^{-s\delta-s}(1+\lambda x^{-\delta})^{-\beta acs-s}[1-(1+\lambda x^{-\delta})^{-c\beta}]^{bs-s}dx \\
&= \frac{(c\beta\lambda\delta)^s}{B^s(a,b)}\int_0^1 \lambda^{-s\delta-s} y^{\frac{-s\delta-s}{\delta}+\beta acs+s-2+\frac{1}{\delta}+1}(1-y^{\beta c})^{sb-s}(1-y)^{s-1+\frac{s-1}{\delta}}dy.
\end{aligned}$$

Using the fact that, for $|\omega| < 1$, $(1-\omega)^{b-1} = \sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(b)}{\Gamma(b-j)j!}\omega^j$, and setting $y = (1+\lambda x^{-\delta})^{-1}$, so that $x^{-\delta} = \frac{y^{-1}-1}{\lambda} = \frac{1-y}{\lambda y}$, and $\lambda\delta x^{-\delta-1}dx = y^{-2}dy$, we obtain

$$\begin{aligned}
\int_0^\infty f^s(x)dx &= \frac{(c\beta\lambda\delta)^s}{B^s(a,b)}\int_0^1 \lambda^{-s\delta-s} y^{\frac{-s\delta-s}{\delta}+\beta acs+s-2+\frac{1}{\delta}+1}(1-y^{\beta c})^{sb-s}(1-y)^{s+\frac{s}{\delta}-\frac{1}{\delta}-1}dy \\
&= \frac{(c\beta\lambda\delta)^s\lambda^{1+\frac{1}{\delta}-s\delta-s}}{B^s(a,b)}\int_0^1 y^{\beta acs-\frac{s}{\delta}+\frac{1}{\delta}-1}(1-y)^{s+\frac{s}{\delta}-\frac{1}{\delta}-1}\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(sb-s+1)y^{\beta cj}}{\Gamma(sb-s+1-j)j!}dy \\
&= \frac{(c\beta\lambda\delta)^s\lambda^{1+\frac{1}{\delta}-s\delta-s}}{B^s(a,b)}\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(sb-s+1)}{\Gamma(sb-s+1-j)j!}B(\beta cj+\beta acs-\tfrac{s}{\delta}+\tfrac{1}{\delta}, s+\tfrac{s}{\delta}-\tfrac{1}{\delta}).
\end{aligned} \tag{31}$$

Therefore, Renyi entropy for the Mc-Dagum distribution is

$$\begin{aligned}
H_R(f) &= \frac{log}{1-s}\Bigg[\frac{(c\beta\lambda\delta)^s\lambda^{1+\frac{1}{\delta}-s\delta-s}}{B^s(a,b)}\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(sb-s+1)}{\Gamma(sb-s+1-j)j!} \\
&\qquad B\left(\beta cj+\beta acs-\frac{s}{\delta}+\frac{1}{\delta}, s+\frac{s}{\delta}-\frac{1}{\delta}\right)\Bigg]
\end{aligned} \tag{32}$$

for $s > 0$ and $s \neq 1$. If $bs - s$ is a positive integer, then the sum in the Renyi entropy stops at $bs - s$.

5.2. **$\tilde{\beta}$-entropy.** We also obtain $\tilde{\beta}$-entropy for the Mc-Dagum distribution. Note that $\tilde{\beta}$-entropy is given by

$$H_{\tilde{\beta}}(f)=\frac{1}{\tilde{\beta}-1}\left[1-\int_0^{\infty} f^{\tilde{\beta}}(x)dx\right], \text{if } \tilde{\beta}\neq 1, \text{ and } \tilde{\beta}>0.$$

If $\tilde{\beta}=1$, then $H_{\tilde{\beta}}(f)=E[-log(f(X))]$ is the Shannon entropy.

For $\tilde{\beta}\neq 1$, and $\tilde{\beta}>0$, $\tilde{\beta}$-entropy for the Mc-Dagum distribution is

$$\begin{aligned} H_{\tilde{\beta}}(f) &= \frac{1}{\tilde{\beta}-1}\Bigg[1-\frac{(c\beta\lambda\delta)^{\tilde{\beta}}\lambda^{1+\frac{1}{\delta}-\tilde{\beta}\delta-\tilde{\beta}}}{B^{\tilde{\beta}}(a,b)}\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\tilde{\beta}b-\tilde{\beta}+1)}{\Gamma(\tilde{\beta}b-\tilde{\beta}+1-j)j!} \\ &\times\; B\left(\beta cj+\beta ac\tilde{\beta}-\frac{\tilde{\beta}}{\delta}+\frac{1}{\delta},\tilde{\beta}+\frac{\tilde{\beta}}{\delta}-\frac{1}{\delta}\right)\Bigg]. \end{aligned}$$

6. Maximum Likelihood Estimates

Let $\Theta=(\lambda,\beta,\delta,a,b,c)^T$. In order to estimate the parameters λ,β,δ,a,b and c of the Mc-Dagum distribution, we use the method of maximum likelihood estimation. Let $x_1,x_2,......,x_n$ be a random sample from $f(x;\lambda,\beta,\delta,a,b,c)$. The log-likelihood function $L(\lambda,\beta,\delta,a,b,c)$ is:

$$\begin{aligned} L(\lambda,\beta,\delta,a,b,c) &= nlog(c)+nlog(\beta)+nlog(\lambda)+nlog(\delta)-nlogB(a,b) \\ &-(\delta+1)\sum_{i=1}^{n}logx_i-(\beta ac+1)\sum_{i=1}^{n}log[1+\lambda x_i^{-\delta}] \\ &+(b-1)\sum_{i=1}^{n}log[1-(1+\lambda x_i^{-\delta})^{-c\beta}]. \end{aligned}$$

Differentiating $L(\lambda,\beta,\delta,a,b,c)$ with respect to each parameter λ,β,δ,a,b and c and setting the result equals to zero, we obtain maximum likelihood estimates. The partial derivatives of L with respect to each parameter or the score function is given by:

$$U_n(\Theta)=\left(\frac{\partial L}{\partial\lambda},\frac{\partial L}{\partial\beta},\frac{\partial L}{\partial\delta},\frac{\partial L}{\partial a},\frac{\partial L}{\partial b},\frac{\partial L}{\partial c}\right), \tag{33}$$

where

(34)

$$\frac{\partial L}{\partial\lambda}=\frac{n}{\lambda}-\beta ac\sum_{i=1}^{n}\left(\frac{x_i^{-\delta}}{1+\lambda x_i^{-\delta}}\right)-\sum_{i=1}^{n}\frac{x_i^{-\delta}}{(1+\lambda x_i^{-\delta})}+(b-1)\sum_{i=1}^{n}\frac{c\beta(1+\lambda x_i^{-\delta})^{-c\beta-1}x_i^{-\delta}}{[1-(1+\lambda x_i^{-\delta})^{-c\beta}]},$$

$$\frac{\partial L}{\partial\beta}=\frac{n}{\beta}-ac\sum_{i=1}^{n}log(1+\lambda x_i^{-\delta})+c(b-1)\sum_{i=1}^{n}\frac{(1+\lambda x_i^{-\delta})^{-c\beta}log(1+\lambda x_i^{-\delta})}{[1-(1+\lambda x_i^{-\delta})^{-c\beta}]}, \tag{35}$$

$$\begin{aligned} \frac{\partial L}{\partial\delta} &= \frac{n}{\delta}-\sum_{i=1}^{n}log(x_i)+\lambda(\beta ac+1)\sum_{i=1}^{n}\frac{x_i^{-\delta}log(x_i)}{(1+\lambda x_i^{-\delta})} \\ &-\;\lambda c\beta(b-1)\sum_{i=1}^{n}\frac{x_i^{-\delta}(1+\lambda x_i^{-\delta})^{-c\beta-1}log(x_i)}{[1-(1+\lambda x_i^{-\delta})^{-c\beta}]}, \end{aligned} \tag{36}$$

$$\frac{\partial L}{\partial a} = -n(\psi(a) - \psi(a+b)) - \beta c \sum_{i=1}^{n} log(1+\lambda x_i^{-\delta}), \tag{37}$$

$$\frac{\partial L}{\partial b} = -n[\psi(b) - \psi(a+b)] + \sum_{i=1}^{n} log[1-(1+\lambda x_i^{-\delta})^{-c\beta}], \tag{38}$$

where $\psi(.)$ is digamma function $\psi(x) = \frac{d}{dx} log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, and

$$\frac{\partial L}{\partial c} = \frac{n}{c} - \beta a \sum_{i=1}^{n} log(1+\lambda x_i^{-\delta}) + (b-1)\beta \sum_{i=1}^{n} \frac{(1+\lambda x_i^{-\delta})^{-c\beta} log(1+\lambda x_i^{-\delta})}{[1-(1+\lambda x_i^{-\delta})^{-c\beta}]}. \tag{39}$$

The MLE of the parameters λ, β, δ, a, b and c, say $\hat{\lambda}$, $\hat{\beta}$, $\hat{\delta}$, $\hat{a}$, $\hat{b}$ and $\hat{c}$ are obtained by solving the following equations, $\frac{\partial L}{\partial \lambda} = \frac{\partial L}{\partial \beta} = \frac{\partial L}{\partial \delta} = \frac{\partial L}{\partial a} = \frac{\partial L}{\partial b} = \frac{\partial L}{\partial c} = 0$. There is no closed form solution to these equations, so numerical technique such as Newton Raphson method must be applied.

7. Fishers Information Matrix

To obtain the Fishers information matrix (FIM), we derive the second partial derivatives and cross partial derivatives with respect to each parameter λ, β, δ, a, b, and c as follows:

$$\frac{\partial^2 L}{\partial \lambda^2} = \frac{-n}{\lambda^2} + (\beta ac+1) \sum_{i=1}^{n} \frac{x_i^{-2\delta}}{A_i^2} + (b-1) \sum_{i=1}^{n} \frac{c\beta x_i^{-2\delta} A_i^{-c\beta-2}[A_i^{-c\beta} - c\beta - 1]}{[1-A_i^{-c\beta}]^2}, \tag{40}$$

where, $A_i = (1+\lambda x_i^{-\delta})$,

$$\frac{\partial^2 L}{\partial \lambda \partial \beta} = -ac \sum_{i=1}^{n} \frac{x_i^{-\delta}}{A_i} + (b-1) \sum_{i=1}^{n} x_i^{-\delta} c A_i^{-c\beta-1} \frac{[c\beta log(A_i) - 1 + A_i^{-c\beta}]}{[1-A_i^{-c\beta}]^2}, \tag{41}$$

$$\frac{\partial^2 L}{\partial \lambda \partial \delta} = (-\beta ac - 1) \sum_{i=1}^{n} \frac{x_i^{-\delta} log(x_i)}{A_i^2} + (b-1) \sum_{i=1}^{n} c\beta x_i^{-\delta} A_i^{-c\beta-1} log(x_i) B_i, \tag{42}$$

where $B_i = \frac{1-\lambda x_i^{-\delta} c\beta A_i^{-1} - \lambda x_i^{-\delta} A_i^{-1} + \lambda x_i^{-\delta} c\beta A_i^{-c\beta-1} - (1+\lambda x_i^{-\delta})^{-c\beta}}{[1-A_i^{-c\beta}]^2}$,

$$\frac{\partial^2 L}{\partial \lambda \partial a} = -\beta c \sum_{i=1}^{n} \frac{x_i^{-\delta}}{(1+\lambda x_i^{-\delta})}, \tag{43}$$

$$\frac{\partial^2 L}{\partial \lambda \partial b} = \sum_{i=1}^{n} \frac{c\beta(1+\lambda x_i^{-\delta})^{-c\beta-1} x_i^{-\delta}}{[1-(1+\lambda x_i^{-\delta})^{-c\beta}]}, \tag{44}$$

$$\frac{\partial^2 L}{\partial \lambda \partial c} = -\beta a \sum_{i=1}^{n} \frac{x_i^{-\delta}}{A_i} + (b-1) \sum_{i=1}^{n} \frac{\beta x_i^{-\delta} A_i^{-c\beta-1}[c\beta log A_i - 1 + A_i^{-c\beta}]}{[1-A_i^{-c\beta}]^2}, \tag{45}$$

$$\frac{\partial^2 L}{\partial \beta^2} = \frac{-n}{\beta^2} + c^2(b-1) \sum_{i=1}^{n} \frac{(1+\lambda x_i^{-\delta})^{-c\beta}[log(1+\lambda x_i^{-\delta})]^2}{[1-(1+\lambda x_i^{-\delta})^{-c\beta}]^2}, \tag{46}$$

$$\frac{\partial^2 L}{\partial \beta \partial \delta} = ac \sum_{i=1}^{n} C_i + (b-1) \sum_{i=1}^{n} \frac{\lambda c A_i^{-c\beta-1} x_i^{-\delta} log x_i [1 - c\beta log A_i - A_i^{-c\beta}]}{[1-A_i^{-c\beta}]^2}, \tag{47}$$

where $C_i = \frac{\lambda x_i^{-\delta} log(x_i)}{A_i}$,

$$\frac{\partial^2 L}{\partial\beta\partial a} = -c\sum_{i=1}^{n} log(1+\lambda x_i^{-\delta}), \tag{48}$$

$$\frac{\partial^2 L}{\partial\beta\partial b} = \sum_{i=1}^{n} \frac{c(1+\lambda x_i^{-\delta})^{-c\beta} log(1+\lambda x_i^{-\delta})}{[1-(1+\lambda x_i^{-\delta})^{-c\beta}]}, \tag{49}$$

$$\frac{\partial^2 L}{\partial\beta\partial c} = -a\sum_{i=1}^{n} logA_i + (b-1)\sum_{i=1}^{n} \frac{A_i^{-c\beta} logA_i[c\beta logA_i + A_i^{-c\beta} - 1]}{[1-A_i^{-c\beta}]^2}, \tag{50}$$

$$\frac{\partial^2 L}{\partial\delta^2} = \frac{-n}{\delta^2} + \lambda(\beta ac+1)\sum_{i=1}^{n} x_i^{-\delta}(logx_i)^2 A_i^{-c\beta-1} D_i \tag{51}$$

where $D_i = \frac{[1-\lambda c\beta x_i^{-\delta}(1+\lambda x_i^{-\delta})^{-1} - \lambda x_i^{-\delta}(1+\lambda x_i^{-\delta})^{-1} - (1+\lambda x_i^{-\delta})^{-c\beta} + \lambda x_i^{-\delta}(1+\lambda x_i^{-\delta})^{-c\beta-1}]}{[1-(1+\lambda x_i^{-\delta})^{-c\beta}]^2}$.

Also,

$$\frac{\partial^2 L}{\partial\delta\partial a} = \lambda\beta c\sum_{i=1}^{n} \frac{x_i^{-\delta} logx_i}{(1+\lambda x_i^{-\delta})}, \tag{52}$$

$$\frac{\partial^2 L}{\partial\delta\partial b} = -\lambda c\beta\sum_{i=1}^{n} \frac{x_i^{-\delta}(1+\lambda x_i^{-\delta})^{-c\beta-1} logx_i}{[1-(1+\lambda x_i^{-\delta})^{-c\beta}]}, \tag{53}$$

$$\frac{\partial^2 L}{\partial\delta\partial c} = \lambda\beta c\sum_{i=1}^{n} F_i - \lambda\beta(b-1)\sum_{i=1}^{n} \frac{A_i^{-c\beta-1} x_i^{-\delta} logx_i[c\beta logA_i + A_i^{-c\beta} - 1]}{[1-A_i^{-c\beta}]^2}, \tag{54}$$

where $F_i = \frac{x_i^{-\delta} logx_i}{(1+\lambda x_i^{-\delta})}$,

$$\frac{\partial^2 L}{\partial a^2} = n\left[(\psi(a+b))^2 - \frac{\Gamma''(a+b)}{\Gamma(a+b)} - (\psi(a))^2 + \frac{\Gamma''(a)}{\Gamma(a)}\right], \tag{55}$$

$$\frac{\partial^2 L}{\partial a\partial b} = n\left[(\psi(a+b))^2 - \frac{\Gamma''(a+b)}{\Gamma(a+b)}\right], \tag{56}$$

$$\frac{\partial^2 L}{\partial a\partial c} = -\beta\sum_{i=1}^{n} log(1+\lambda x_i^{-\delta}), \tag{57}$$

$$\frac{\partial^2 L}{\partial a^2} = n\left[(\psi(a+b))^2 - \frac{\Gamma''(a+b)}{\Gamma(a+b)} - (\psi(b))^2 + \frac{\Gamma''(b)}{\Gamma(b)}\right], \tag{58}$$

$$\frac{\partial^2 L}{\partial b\partial c} = \sum_{i=1}^{n} \frac{\beta(1+\lambda x_i^{-\delta})^{-c\beta} log(1+\lambda x_i^{-\delta})}{[1-(1+\lambda x_i^{-\delta})^{-c\beta}]}, \tag{59}$$

and

$$\frac{\partial^2 L}{\partial c^2} = \frac{-n}{c^2} + \beta(b-1)\sum_{i=1}^{n} \frac{\beta(1+\lambda x_i^{-\delta})^{-c\beta}[log(1+\lambda x_i^{-\delta})]^2}{[1-(1+\lambda x_i^{-\delta})^{-c\beta}]^2}. \tag{60}$$

Fisher information matrix (FIM) for the Mc-D distribution is:

$$I(\lambda,\beta,\delta,a,b,c)=\begin{pmatrix} I_{\lambda\lambda} & I_{\lambda\beta} & I_{\lambda\delta} & I_{\lambda a} & I_{\lambda b} & I_{\lambda c}\\ I_{\beta\lambda} & I_{\beta\beta} & I_{\beta\delta} & I_{\beta a} & I_{\beta b} & I_{\beta c}\\ I_{\delta\lambda} & I_{\delta\beta} & I_{\delta\delta} & I_{\delta a} & I_{\delta b} & I_{\delta c}\\ I_{a\lambda} & I_{a\beta} & I_{a\delta} & I_{aa} & I_{ab} & I_{ac}\\ I_{b\lambda} & I_{b\beta} & I_{b\delta} & I_{ba} & I_{bb} & I_{bc}\\ I_{c\lambda} & I_{c\beta} & I_{c\delta} & I_{ca} & I_{cb} & I_{cc} \end{pmatrix},$$

where $I_{\lambda\lambda}=-E\left[\frac{\partial^2 L}{\partial\lambda^2}\right]$,........,$I_{cc}=-E\left[\frac{\partial^2 L}{\partial c^2}\right]$. The elements of the 6 X 6 matrix $I(\lambda,\beta,\delta,a,b,c)$ can be approximated by the elements of the observed information matrix, where

$$I_{ij}(\theta)=-E\left[\frac{\partial^2 L}{\partial\theta_i\partial\theta_j}\right]\approx\frac{-\partial^2 L}{\partial\theta_i\partial\theta_j}=\mathbf{J_n}(\boldsymbol{\Theta}).$$

7.1. **Asymptotic Confidence Intervals.** In this section, we present the asymptotic confidence intervals for the parameters of the Mc-Dagum distribution. The expectations in the Fisher Information Matrix (FIM) can be obtained numerically. Applying the usual large sample approximation, MLE of Θ, that is, $\hat{\Theta}$ is approximately $N_6(\Theta, I_n^{-1}(\Theta))$, where $I_n(\Theta)$ is the 6X6 observed information matrix. Under the condition that the parameters are in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\hat{\Theta}-\Theta)$ is $N_6(\Theta, I^{-1}(\Theta))$, where $I(\Theta)=\lim_{n\to\infty} n^{-1}I_n(\Theta)$ and

$$I_n(\Theta)=n\begin{pmatrix} I_{\lambda\lambda} & I_{\lambda\beta} & I_{\lambda\delta} & I_{\lambda a} & I_{\lambda b} & I_{\lambda c}\\ I_{\beta\lambda} & I_{\beta\beta} & I_{\beta\delta} & I_{\beta a} & I_{\beta b} & I_{\beta c}\\ I_{\delta\lambda} & I_{\delta\beta} & I_{\delta\delta} & I_{\delta a} & I_{\delta b} & I_{\delta c}\\ I_{a\lambda} & I_{a\beta} & I_{a\delta} & I_{aa} & I_{ab} & I_{ac}\\ I_{b\lambda} & I_{b\beta} & I_{b\delta} & I_{ba} & I_{bb} & I_{bc}\\ I_{c\lambda} & I_{c\beta} & I_{c\delta} & I_{ca} & I_{cb} & I_{cc} \end{pmatrix}.$$

The multivariate normal distribution with mean vector $(0,0,0,0,0,0)^T$ and covariance matrix $I_n(\Theta)$ can be used to construct confidence intervals for the model parameters. That is, the approximate $100(1-\eta)\%$ two-sided confidence intervals for λ, β, δ, a, b and c are given by:

$$\widehat{\lambda}\pm Z_{\frac{\eta}{2}}\sqrt{I_{\lambda\lambda}^{-1}(\widehat{\Theta})},\quad \widehat{\beta}\pm Z_{\frac{\eta}{2}}\sqrt{I_{\beta\beta}^{-1}(\widehat{\Theta})},\quad \widehat{\delta}\pm Z_{\frac{\eta}{2}}\sqrt{I_{\delta\delta}^{-1}(\widehat{\Theta})},\quad \widehat{a}\pm Z_{\frac{\eta}{2}}\sqrt{I_{aa}^{-1}(\widehat{\Theta})},$$

$$\widehat{b}\pm Z_{\frac{\eta}{2}}\sqrt{I_{bb}^{-1}(\widehat{\Theta})},\quad \text{and}\quad \widehat{c}\pm Z_{\frac{\eta}{2}}\sqrt{I_{cc}^{-1}(\widehat{\Theta})},$$

respectively, where $Z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}^{th}$ percentile of a standard normal distribution.

We can use the likelihood ratio (LR) test to compare the fit of the Mc-D distribution with its sub-models for a given data set. Note that, to test $b=c=1$, the LR statistic is

$$\omega=2[\ln(L(\hat{\beta},\hat{\delta},\hat{\lambda},\hat{a},\hat{b},\hat{c}))-\ln(L(\tilde{\beta},\tilde{\delta},\tilde{\lambda},\tilde{a},1,1))],$$

where $\hat{\beta}$, $\hat{\delta}$, $\hat{\lambda}$, $\hat{a}$, $\hat{b}$, and $\hat{c}$, are the unrestricted estimates, and $\tilde{\beta}$, $\tilde{\delta}$, $\tilde{\lambda}$, and $\tilde{a}$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega>\chi^2_d$, where χ^2_d denote the upper $100d\%$ point of the χ^2 distribution with 2 degrees of freedom.

TABLE 1. Estimation of models for failure times data.

	Estimates						Statistics			
Distribution	β	δ	λ	a	b	c	$-2\ln(L)$	AIC	AICC	BIC
Mc-D	4.1652	2.7600	106505	0.01904	45431	4.1652	126.0	138.0	148.5	142.3
	(5.3470)	(0.3228)	(2.966E-6)	(0.01999)	(0.04933)	(5.3470)				
B-D	7.2221	8.8376	3.5995	0.6595	0.04258	-	147.1	157.1	163.8	160.6
	(2.9051)	(0.5793)	(6.9162)	(0.8025)	(0.01171)	-				
B-BIII	0.8308	0.09752	-	290.58	172.49	-	131.3	139.3	143.3	142.1
	(0.03511)	(0.01684)	-	(0.2349)	(0.1349)	-				
D	1.1568	1.5295	77.9568	-	-	-	131.6	137.6	139.8	139.7
	(0.6532)	(0.3450)	(123.96)	-	-	-				
BIII	1.3821	1.2237	-	-	-	-	136.0	140.0	141.0	141.5
	(0.04983)	(0.03518)	-	-	-	-				
Mc-W	-	8.8539	0.01463	0.006759	362.23	13.2979	126.0	136.0	142.5	139.5
		(0.4082)	(0.001952)	(0.009667)	(12.1191)	(18.8857)				

8. Numerical Application: Mc-Dagum and Sub-Distributions

In this section, applications based on real data, as well as comparison of the Mc-Dagum distribution with its sub-models are presented. We provide examples to illustrate the flexibility of the Mc-Dagum distribution in contrast to other models for data modeling. These data sets are modeled by the Mc-D distribution and compared with the corresponding sub-models, including the Dagum distribution. The first data set (Lawless [12]) represent the failure times, in minutes, of 15 electronic components in an accelerated life test and they are as follows: 1.4, 5.1, 6.3, 10.8, 12.1, 18.5, 19.7, 22.2, 23, 30.6, 37.3, 46.3, 53.9, 59.8, 66.2. The second data set presented in Tables 2 and 3 represents the salaries (in dollars) of 818 professional baseball players for the year 2008. We fit Mc-Dagum(Mc-D), beta-Dagum (B-D), beta-Burr III (B-BIII), Dagum (D) and McDonald Weibull (Mc-W) distributions to these data using the method of maximum likelihood estimation. The MLEs of the parameters (with standard errors in parenthesis), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) for the fitted models are presented in Tables 1 and 4. The Mc-Weibull pdf [7] is given by

$$f(x;\lambda,\delta,a,b,c)=\frac{c\delta\lambda^{\delta}x^{\delta-1}}{B(a,b)}e^{-(\lambda x)^{\delta}}(1-e^{-(\lambda x)^{\delta}})^{ac-1}[1-(1-e^{-(\lambda x)^{\delta}})^{c}]^{b-1}, \quad (61)$$

for $\lambda,\delta,a,b,c>0$ and $x>0$.

The maximum likelihood estimates (MLEs) of the parameters are computed by maximizing the objective function via the subroutine NLMIXED in SAS. The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic, Akaike Information Criterion, $AIC = 2k - 2\ln(L)$, Bayesian Information Criterion, $BIC = k\ln(n) - 2\ln(L)$, and Consistent Akaike Information Criterion, $AICC = AIC + 2\frac{k(k+1)}{n-k-1}$, where $L = L(\hat{\Theta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and k is the number of estimated parameters are presented in Tables 1 and 4 for the Mc-D distribution and its sub-distributions.

For the failure time data set, the likelihood ratio statistics for the test of the hypotheses H_0 : B-D against H_a : Mc-D and $H_0 : BIII$ against $H_a : Mc-D$ are 21.1 (p-value<0.00001) and 10 (p-value <0.05), respectively. Also, the likelihood ratio statistic for the test of H_0 : B-BIII against H_a : Mc-D is 5.3 (p-value=0.07065). Consequently, we reject the null hypothesis in favor of the Mc-D distribution and conclude that the Mc-D distributions are significantly better than the Dagum and Beta-Dagum distributions based on the LR statistic. The values of the statistics

TABLE 2. Baseball Player Salary ($\times 10^6$)

0.403	1.75	6	0.4345	4.956237	5.5	0.75	0.475	1.5
11.666666	13	0.4186	2.5	0.406	13.1	1.775	2.7875	0.4
0.8	6.25	0.404	1	1.0325	0.42245	9	4.05	0.4
8.75	1.75	5.9	1.75	4	0.4551	3.125	0.975	5.5
1.5	5	1.5	1.7	10	15	0.4073	1.4	8
6.25	0.441	3.65	2	0.800002	33	0.4	1.98125	0.424
0.5	1.5	0.4363	3.5	1	15.285714	1.25	3.666666	0.75
0.401	5.5	0.4142	0.4275	0.403	5.4	0.4115	7	0.4
7.5	0.4	1.95	19	0.4	20.625	0.5	0.675	0.452
3.05	5	4.766666	5.5	7	0.432975	0.4044	8.25	0.445
3.5	4	0.4275	0.75	0.414	5	0.4324	0.4333	2.8
0.425	6.25	10	2.3	6.925	0.4	0.4309	1.255	0.475
0.7125	7	10	0.75	4.65	0.4	0.4	0.4337	0.425
0.43	1.1	7.05	14	3.25	0.405	0.41	0.75	0.5
0.4115	9	0.415	0.439	6	0.41	11	3.25	2.95
2.535	0.43	1.625	0.61	0.95	0.41	0.675	0.4104	5
2.525	12.5	1.055	1.5	5	4	1.6875	1.000018	0.4
0.4225	0.8	9.375	2.6	4.75	0.4	0.4	6	3.325
0.4	4.2	0.75	0.449	1.6625	0.42	0.4005	0.55	
1.45	0.4215	2	0.5	0.4	11.5	4.625	2.1	
3.6	7	2	14.811414	0.4	0.4	3.5	1.3	
0.4225	1.625	0.401	8	0.426	5.3	1	1.29	
2	0.475	3.75	0.425	4.25	2.8	0.404	8	
5	0.4075	0.8	0.4245	0.415	0.41	2.9	4.445	
0.412	3	5.5	1.2	0.4075	0.41	0.40175	0.4028	
6.5	13.25	16.6	1.4	12.75	3.9	2.75	0.4139	
1.95	13	0.415	2.9	0.447	5	0.401	1.5	
2.5	0.575	0.75	0.421	2.45	0.41	8	1.35	
0.4375	0.45	9.875	0.4155	1.3	0.41	0.44	0.4301	
0.4325	2.2	0.402	0.411	0.75	1	0.65	0.435	
0.42	0.4	0.401	6.083333	0.44	4	4.75	3.2	
0.425	16.65	0.408	2.25	0.45	13.5	0.4	0.4159	
1.2375	17	0.75	1.3	0.422	0.4	0.41	0.4	
0.425	0.575	0.403	3.6	0.4125	0.4	0.475	3.75	
3.375	0.5	0.407	4.25	0.4275	3.3	6.5	0.75	
1	3.5	0.415	0.457	1.7	0.41	8	6.2	
3.45	18.75	0.404	3.75	2.5	0.41	0.4017	3.06	
15.5	0.44	4.5	12	7.666666	1.1	9.6	0.40807	
1.85	1.3	3.75	3.275	0.525	0.405	0.455	0.40662	
2.825	14	1	4	0.4	5.475	0.40175	0.41	
10	0.44	0.4	11.4	0.4375	0.8	0.40125	0.43468	
0.45	1	12.5	2.7	1.475	0.5	0.4	1.615	
8.333666	10	0.4	0.425	0.4661	0.65	1	1	
2.885	0.52	14.383049	0.405	1.15	0.46	0.401	0.555	
15	6	1	1.8	2.75	1.15	0.404	0.4055	
3.7	11.5	0.435	1	0.43	1.635	0.48	0.401	
0.405	0.41	3.5	3.575	10.5	2	9.6	0.8	
0.41	0.75	10	0.437	0.4	5	18.5	0.5	
0.8	0.401	6.3	0.64	11.6	4.35	0.419	3.2	
0.4	5.6	2.2	5	11.25	0.405	0.405	0.40473	

AIC, AICC and BIC shows that the Mc-D distribution gives smaller values than B-D, B-BIII, D, BIII, and compares favorably with the Mc-Weibull distribution.

For the baseball player salary data set, the likelihood ratio statistics for the test of the hypotheses $H_0 : D$ against H_a : Mc-D and H_0 : B-D against H_a : Mc-D are 519.4 (p-value$<$ 0.00001) and 14.6 (p-value $<$ 0.0002), respectively. Consequently, we reject the null hypothesis in favor of the Mc-D distribution and conclude that the Mc-D distribution is significantly better than the Dagum and beta-Dagum distributions based on the LR statistic. A closer look at the values of the statistics AIC, AICC and BIC shows that the Mc-D distribution gives smaller values than B-D, B-BIII, D, BIII, and Mc-Weibull distributions.

Table 3. (Continued)Baseball Player Salary ($\times 10^6$)

0.415	12	2.8	0.4	0.44	15	9.5	0.65
1.4	0.4025	0.403	0.41	0.43	7.166666	7.75	12.868892
0.4	4.5	4.25	0.46	4	12	13.4	0.41482
0.4	2.65	0.4375	0.403	0.95	2.333333	2.3	0.41176
6.35	0.45	18.971596	10	0.44	6.5	1.4	12
11.5	0.4	0.4	5.775	0.435	12.083333	0.8225	0.41631
0.41	6.25	4.6	8.5	1.152	2.5	2.05	0.41089
0.4	0.55	2.095	15	1.15	8.5	0.413	0.401
7.166666	1.225	0.75	18	19.243682	4.25	0.405	1
0.405	0.405	7	1.6	6.25	0.475	2	1.85
4	13	2.7	0.401	2.625	1.625	0.455	13.054526
0.65	1.325	0.825	0.465	2.8	0.835	3.8	2.5
1.5	0.55	1.3	10	2	11.285714	0.4	0.6
0.445	10.125	2.275	0.405	12	3.125	0.405	2.4
8	0.4175	0.75	0.45	1.6125	2.5	7.666666	0.75
2.4	0.42	3.675	10.4	0.471	0.4	0.4	0.4087
0.42	12.125	10	1.1	2.6	0.4085	0.4	3.75
0.435	0.418	0.735	0.435	0.401	2.425	2.25	1.45
3.35	0.4	1.3	2	0.406	0.402	0.418	14.25
0.8	2.425	0.4	3.665	0.575	0.4015	0.425	2.59
8	5.375	0.4	1.1	0.4	2.15	0.42	0.4139
0.41	2	0.5	3.25	2.2375	2.2	12.25	0.64
0.85	0.4	3.5	3.8	12	0.4015	18	0.4118
8	14	0.4	0.475	6	2.3	0.5	0.4144
0.41	8.5	1.9	5	2.25	0.825	0.4	0.4052
2.4	0.4	0.42	4.75	0.925	1.5	9.85	1.9
2.75	0.4	0.95	0.465	6.125	0.4	2.825	0.4194
0.4	1.5	2.25	1	9.166666	0.4015	0.4	0.85
5	0.418	1.4	0.475	18.876139	7.05	13.302583	7.95
4.5	0.42	0.66	5	4.9	0.4135	0.825	0.4037
2.5	5.1875	0.41	1.825	14	2.5	2.5	0.4023
1.5	2	0.4	0.405	0.4095	2.5	0.4	6.4
0.5	2.25	0.4	3.1	1.7	0.75	0.4	11.625
7.8	0.44	2.4625	7.5	10.5	0.4115	12.137	0.4
11.166666	0.4375	2.4	3.364877	7.75	0.4145	6.5	12
0.4	3.5	0.4125	0.467	0.403075	0.4135	0.405	1.1
0.476	0.4161	0.44	0.404	1.25	6.25	0.95	0.4014
14	3.35	0.4	0.437	16.5	3.2	7.4375	1.015
0.4495	0.4167	0.404	12.433333	1.4	1.875	3.7	4.6875
0.4	2.9375	0.75	0.465	6	0.4	0.411	1.6
0.55	2.5	5.5	1.25	0.432575	7.4	0.5	2
1.5	0.4203	2.225	3.9	0.4033	1.3	3.3125	1.9
0.4	0.4	0.4	0.40075	13	0.4148	0.4	2.6
1	5.5	5.35	2.35	0.414	0.4299	7.5	0.431
1.35	4	0.4	10	21.6	1.255	14.427326	0.4155
12.5	0.4214	0.4	23.854494	3.75	0.75	0.4	8
0.414	0.4461	1.55	15.217401	13	0.4	1	0.4155
9.25	11.5	14.5	0.401	2.125	0.75	0.403	0.4
0.4155	0.4038	1.25	0.4	6.55	0.4	0.43	8
8.333333	3	0.75	0.4025	0.4	0.85	0.65	0.42

Table 4. Estimation of models for baseball player salary data.

	Estimates						Statistics			
Distribution	β	δ	λ	a	b	c	$-2\ln(L)$	AIC	AICC	BIC
Mc-D	47.9710	8.0983	0.000220	0.1053	0.06001	0.8718	2706.2	2718.2	2718.3	2746.4
	(9.9363)	(0.05615)	(0.000014)	(0.01043)	(0.002683)	(0.1580)				
B-D	24.7269	8.2030	0.000317	0.1196	0.07237	-	2720.8	2730.8	2730.8	2754.3
	(1.0055)	(0.02546)	(0.000041)	(0.01180)	(0.003253)	-				
B-BIII	8.4648	1.2126	-	0.1646	1.0218	-	3367.3	3375.3	3375.3	3394.1
	(2.4174)	(0.06022)	-	(0.06231)	(0.09221)	-				
D	70.078	1.0301	0.01163	-	-	-	3225.6	3231.6	3231.6	3245.7
	(34.9306)	(0.03011)	(0.005860)	-	-	-				
BIII	1.3821	1.2237	-	-	-	-	3367.3	3371.3	3371.3	3380.7
	(0.04983)	(0.03518)	-	-	-	-				
Mc-W	-	0.2562	68.7906	0.01455	4.9316	1315.50	3300.8	3310.8	3310.8	3334.3
		(0.01189)	(26.4644)	(0.003375)	(0.1424)	(354.12)				

9. Concluding Remarks

We have proposed and presented results on the mathematical and statistical properties of the Mc-Dagum distribution. This class of distribution contains a fairly large number of distributions with potential applications to a wide area of probability and statistics including income and lifetime data analysis. Properties of the class of Mc-Dagum distributions including the pdf, cdf, moment, hazard function, reverse hazard function, inequality measures including Lorenz, Bonferroni and Zenga curves, Fisher information, Renyi entropy and β-entropy are derived. Estimation of the parameters of the models and applications to illustrate the usefulness of the distribution are also presented.

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